Explicit Minimal Free Resolutions of Fiber Products

Claim: Under certain "nice" conditions, we can minimally resolve $\frac{R}{\langle I', IJ, J'\rangle} \cong \frac{R}{I'+J} \times_{\frac{R}{I+J}} \frac{R}{I+J'}$ over R.

Motivation: Fiber Products are Tor-friendly (NSW '17)

- minimal free res'l of modules over fiber product
- other classes of rings shown to be Tor-friendly using DG-methods (AINSW '19, '20)

  - need minimal resolutions with DGA structure

Fiber Products:

Consider $S = \dfrac{k[[\underline{x}]]}{I'} \xrightarrow{\pi_S} k \xleftarrow{\pi_T} \dfrac{k[[\underline{y}]]}{J'} = T$

Fiber product: $S \times_k T = \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}$.

$$S \times_k T \cong \frac{k[[\underline{x}, \underline{y}]]}{\langle I', \underline{x}\,\underline{y}, J' \rangle}$$

Goal Today: Minimally resolve

$$\frac{k[[x_1, x_2, y_1, y_2, y_3]]}{\langle x_1^2, x_1x_2, \underline{x}\,\underline{y}, y_2y_3, y_1y_3, y_1y_2 \rangle} \cong \frac{k[[x_1, x_2]]}{\langle x_1^2, x_1x_2 \rangle} \times_k \frac{k[[y_1, y_2, y_3]]}{\langle y_2y_3, y_1y_3, y_1y_2 \rangle}$$

over $R = k[[x_1, x_2, y_1, y_2, y_3]]$.

Step 1: $\dfrac{R}{\langle \underline{x}\,\underline{y} \rangle} \cong k[[\underline{x}]] \times_k k[[\underline{y}]]$ over R.

- Polynomial case done by Visscher '06

Notation:

- $\mathcal{X} = K^R(\underline{x})$: $0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R^2 \xrightarrow{(x_1 \;\; x_2)} R \longrightarrow 0$

- $\mathcal{Y} = K^R(\underline{y})$:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}} R^3 \xrightarrow{(y_1 \;\; y_2 \;\; y_3)} R \rightarrow 0$$

Thm (V'21, -):

The construction

$$(\mathcal{X} * \mathcal{Y})_i = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{i+1} & i \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & i = 0 \end{cases}, \qquad \partial_i^{\mathcal{X}*\mathcal{Y}} = \begin{cases} \partial_{i+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & i \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & i = 1 \end{cases}$$

is a minimal resolution of $\dfrac{R}{\langle \underline{x}\,\underline{y} \rangle}$.

Example!

$$0 \to R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} R^9 \xrightarrow{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}} R^6 \xrightarrow{\substack{\underline{x}\,\underline{y} \\ (x_1y_1 \;\; x_1y_2 \;\; x_1y_3 \;\; x_2y_1 \;\; x_2y_2 \;\; x_2y_3)}} R \to 0$$

Step 2:

Resolve $\dfrac{R}{\langle x_1^2,\, x_1x_2,\, \underline{x}\,y_-\rangle} \cong \dfrac{\Bbbk[[x_1,x_2]]}{(x_1^2,\, x_1x_2)} \times_{\Bbbk} \Bbbk[[y_-]]$ over R

Strategy: $0 \to I \cdot \dfrac{R}{\langle \underline{x}\,y_-\rangle} \longrightarrow \dfrac{R}{\langle \underline{x}\,y_-\rangle} \longrightarrow \dfrac{R}{\langle I,\, \underline{x}\,y_-\rangle} \to 0$

- $\dfrac{R}{\langle \underline{x}\,y_-\rangle}$ resolved by $\mathcal{X} * \mathcal{Y}$
- set $I = (x_1^2,\, x_1x_2)$ and $\mathcal{S}$ the min res'n of R/I
- $I \cdot \dfrac{R}{\langle \underline{x}\,y_-\rangle} \cong I \otimes_R \dfrac{R}{\langle y_-\rangle}$ resolved by $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$
- Need $\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$

Build Φ from $\phi : \mathcal{S} \to \mathcal{X}$:

$$\begin{tikzcd}
\mathcal{S} : \arrow[d, "\phi"] & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"] \arrow[d, "x_1"] & R^2 \arrow[r, "\begin{pmatrix} x_1^2 & x_1 x_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix}"] & R \arrow[r] \arrow[d, "1"] & 0 \\
\mathcal{X} & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"'] & R^2 \arrow[r, "\begin{pmatrix} x_1 & x_2 \end{pmatrix}"'] & R \arrow[r] & 0
\end{tikzcd}$$

Define $\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$

$$\Phi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha| + |\beta|} \phi(\alpha) * \beta & |\beta| > 0 \\ \partial_1^{\mathcal{S}}(\alpha) * \beta & |\beta| = 0, |\alpha| = 1 \\ 0 & |\beta| = 0, \ |\alpha| > 0 \end{cases}$$

Example:

$$
\begin{tikzcd}[ampersand replacement=\&]
\Sigma^{-1}(\mathcal{X}_{\geq 1}\otimes\mathcal{Y}): \arrow[d, "\Phi"] \& 0 \arrow[r] \& R \arrow[r, "{\begin{pmatrix} -y_3 \\ y_2 \\ -y_1 \\ x_2 \\ -x_1 \end{pmatrix}}"] \arrow[d, "-x_1"] \& R^5 \arrow[r, "{\begin{pmatrix} y_2 & y_3 & 0 & 0 & 0 \\ -y_1 & 0 & y_3 & 0 & 0 \\ 0 & -y_1 & -y_2 & 0 & 0 \\ x_2 & 0 & 0 & y_3 & 0 \\ 0 & x_2 & 0 & -y_2 & 0 \\ 0 & 0 & x_2 & y_1 & 0 \\ -x_1 & 0 & 0 & 0 & y_3 \\ 0 & -x_1 & 0 & 0 & -y_2 \\ 0 & 0 & -x_1 & 0 & y_1 \end{pmatrix}}"] \arrow[d, "x_1 I_5"] \& R^9 \arrow[r, "{\begin{pmatrix} -y_1 & -y_2 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & -y_2 & -y_3 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & y_1 & 0 & -y_3 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & y_1 & y_2 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & -y_2 & -y_3 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & y_1 & 0 & -y_3 \\ 0 & 0 & -x_1 & 0 & 0 & 0 & 0 & y_1 & y_2 \end{pmatrix}}"] \arrow[d, "-x_1 I_9"] \& R^7 \arrow[r, "{\begin{pmatrix} x_2 & y_1 & y_2 & y_3 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & y_1 & y_2 & y_3 \end{pmatrix}}"] \arrow[d, "{\left(\begin{array}{c|c} 0 \\ 0 \\ 0 & x_1 I_6 \\ 0 \\ 0 \\ 0 \end{array}\right)}"] \& R^2 \arrow[r] \arrow[d, "{\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}}"] \& 0 \\
\mathcal{X}\otimes\mathcal{Y}: \& 0 \arrow[r] \& R \arrow[r, swap, "{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}}"] \& R^5 \arrow[r, swap, "{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}}"] \& R^9 \arrow[r, swap, "{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}}"] \& R^6 \arrow[r, swap, "{\begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_2y_1 & x_2y_2 & x_2y_3 \end{pmatrix}}"] \& R \arrow[r] \& 0
\end{tikzcd}
$$

Thm: Cone(Φ) is the desired min resolution

Step 3:

Resolve $\dfrac{R}{\langle \underline{x}\,\underline{y}_-,\ y_2y_3,\ y_1y_3,\ y_1y_2\rangle} \cong k[\![\underline{x}]\!] \times_k \dfrac{k[\![\underline{y}_-]\!]}{\langle y_2y_3,\ y_1y_3,\ y_1y_2\rangle}$ over R.

Strategy

$$0 \longrightarrow J\cdot\frac{R}{\langle \underline{x}\,\underline{y}_-\rangle} \longrightarrow \frac{R}{\langle \underline{x}\,\underline{y}_-\rangle} \longrightarrow \frac{R}{\langle \underline{x}\,\underline{y}_-,\ J\rangle} \longrightarrow 0$$

- set $J = \langle y_2y_3,\ y_1y_3,\ y_1y_2\rangle$ and $\mathcal{Z}$ the min res'l of R/J
- $J\cdot\dfrac{R}{\langle \underline{x}\,\underline{y}_-\rangle} \cong \dfrac{R}{\langle \underline{x}\rangle} \otimes_R J$ resolved by $\Sigma^{-1}(\mathcal{X}\otimes_R \mathcal{Z}_{\geq 1})$
- Build $\Psi : \Sigma^{-1}(\mathcal{X}\otimes_R \mathcal{Z}_{\geq 1}) \longrightarrow \mathcal{X} * \mathcal{Y}$ from $\psi : \mathcal{Z} \to \mathcal{Y}$

via

$$\Psi(\alpha\otimes\beta) = \begin{cases} (-1)^{|\alpha|+|\beta|-1}\,\alpha * \psi(\beta) & |\alpha| > 0 \\ \alpha * \partial(\beta) & |\alpha| = 0,\ |\beta| = 1 \\ 0 & |\alpha| = 0,\ |\beta| > 1 \end{cases}$$

Example:

$$
\begin{tikzcd}[ampersand replacement=\&]
\mathcal{Z}: \arrow[d, "\psi"] \& 0 \arrow[r] \& R^2 \arrow[r, "{\begin{pmatrix} -y_1 & 0 \\ y_2 & -y_2 \\ 0 & y_3 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} y_3 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}"] \& R^3 \arrow[r, "{(y_2y_3 \;\; y_1y_3 \;\; y_1y_2)}"] \arrow[d, "{\begin{pmatrix} 0 & y_3 & y_2 \\ y_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}"] \& R \arrow[r] \arrow[d, "1"] \& 0 \\
\mathcal{Y}: \& 0 \arrow[r] \& R \arrow[r, "{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}}"'] \& R^3 \arrow[r, "{\begin{pmatrix} -y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}}"'] \& R^3 \arrow[r, "{(y_1 \;\; y_2 \;\; y_3)}"'] \& R \arrow[r] \& 0
\end{tikzcd}
$$

$$
\begin{tikzcd}[ampersand replacement=\&]
\Sigma^{-1}(\mathcal{X}\otimes\mathcal{Z}_{\geq 1}): \arrow[d, "\Psi"] \& 0 \arrow[r] \& R^2 \arrow[r, "{\begin{pmatrix} y_1 & 0 \\ -y_2 & y_2 \\ 0 & -y_3 \\ x_2 & 0 \\ 0 & x_2 \\ -x_1 & 0 \\ 0 & -x_1 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} -y_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}"] \& R^7 \arrow[r, "{\begin{pmatrix} x_2 & 0 & 0 & -y_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & y_2 & -y_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & y_3 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & y_2 & -y_2 \\ 0 & 0 & -x_1 & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & -x_1 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 0 & -x_1 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 0 & y_3 & y_2 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}"] \& R^8 \arrow[r, "{\begin{pmatrix} -x_1 & 0 & 0 & -x_2 & 0 & 0 & y_1 & 0 \\ 0 & -x_1 & 0 & 0 & -x_2 & 0 & -y_2 & -y_2 \\ 0 & 0 & -x_1 & 0 & 0 & -x_2 & 0 & y_3 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 0 & -y_3 & -y_2 & 0 & 0 & 0 & 0 & 0 \\ -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_3 & -y_2 & 0 & 0 \\ 0 & 0 & 0 & -y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}"] \& R^3 \arrow[r] \arrow[d, "{(y_2y_3 \;\; y_1y_3 \;\; y_1y_2)}"] \& 0 \\
\mathcal{X}+\mathcal{Y}: \& 0 \arrow[r] \& R \arrow[r, "{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}}"'] \& R^5 \arrow[r, "{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}}"'] \& R^9 \arrow[r, "{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}}"'] \& R^6 \arrow[r, "{(x_1y_1 \;\; x_1y_2 \;\; x_1y_3 \;\; x_2y_1 \;\; x_2y_2 \;\; x_2y_3)}"'] \& R \arrow[r] \& 0
\end{tikzcd}
$$

Final step! Resolve $\dfrac{k[[x_1,x_2,y_1,y_2,y_3]]}{\langle x_1^2, x_1x_2, x_- y_-, y_2y_3, y_1y_3, y_1y_2\rangle} \cong \dfrac{k[[x_1,x_2]]}{\langle x_1^2, x_1x_2\rangle} \times_k \dfrac{k[[y_1,y_2,y_3]]}{\langle y_2y_3, y_1y_3, y_1y_2\rangle}$

Strategy: "Glue" together $\mathrm{Cone}(\Phi)$ and $\mathrm{Cone}(\Psi)$

$$\begin{tikzcd}
\Sigma^{-1}(\mathcal{X}_{\geq 1}\otimes \mathcal{Y}): \arrow[d, "\Phi"] & 0 \arrow[r] & R \arrow[r] \arrow[d] & R^5 \arrow[r] \arrow[d] & R^9 \arrow[r] \arrow[d] & R^7 \arrow[r] \arrow[d] & R^2 \arrow[r] \arrow[d, "(x_1^2 \;\; x_1x_2)"] & 0 \\
\mathcal{X} * \mathcal{Y}: & 0 \arrow[r] & R \arrow[r] & R^5 \arrow[r] & R^9 \arrow[r] & R^6 \arrow[r, "(x_1y_1 \;\; x_1y_2 \;\; x_1y_3 \;\; x_2y_1 \;\; x_2y_2 \;\; x_2y_3)"'] & R \arrow[r] & 0 \\
\Sigma^{-1}(\mathcal{X}\otimes \mathcal{Z}_{\geq 1}): \arrow[u, "\Psi"] & & 0 \arrow[r] & R^2 \arrow[r] \arrow[u] & R^7 \arrow[r] \arrow[u] & R^8 \arrow[r] \arrow[u] & R^3 \arrow[r] \arrow[u, "(y_2y_3 \;\; y_1y_3 \;\; y_1y_2)"'] & 0
\end{tikzcd}$$

$$0 \longrightarrow \mathcal{X} * \mathcal{Y} \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}} \begin{matrix} \mathrm{Cone}(\Phi) \\ \oplus \\ \mathrm{Cone}(\Psi) \end{matrix} \longrightarrow \mathrm{Cone}(\Phi\ \Psi) \longrightarrow 0$$

Cone$(\Phi\ \Psi)$ minimally resolves our fiber product.

- recover betti numbers
- " " graded betti numbers
- Poincaré series.