DG-Module Resolutions of Certain Fiber Products

Goal today: Show that the minimal resolution of

$$\frac{k[\![x_1,x_2,y_1,y_2,y_3]\!]}{(x_1^2, x_1x_2, x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3)}$$ over $k[\![x_1,x_2,y_1,y_2,y_3]\!]$ is a

DG-module over the minimal resolution of

$$\frac{k[\![x_1,x_2,y_1,y_2,y_3]\!]}{\langle x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3\rangle}$$ over $k[\![x_1,x_2,y_1,y_2,y_3]\!]$.

Motivation: Fiber Products are Tor-friendly (N SW '17)

- minimal free res'l of modules over fiber product

- other classes of rings shown to be Tor-friendly using DG-methods (AINSW '19, '20)
  - need minimal resolutions with DGA structure

Fiber Products: Consider $S = \dfrac{k[\![\underline{x}]\!]}{I} \xrightarrow{\pi_S} k \xleftarrow{\pi_T} \dfrac{k[\![\underline{y}]\!]}{J} = T$

Fiber product: $S \times_k T = \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}$.

$$S \times_k T \cong \frac{k[\![\underline{x}, \underline{y}]\!]}{\langle I, \underline{x}\underline{y}, J \rangle}$$

Step 1: $\dfrac{k[[x_1,x_2,y_1,y_2,y_3]]}{\langle x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3\rangle} \cong k[[\underline{x}]] \times_k k[[\underline{y}]]$ over $k[[x_1,x_2,y_1,y_2,y_3]]$.

- Polynomial case done by Visscher '06
- Unpublished paper by Sköldberg '16 claims DGA

Notation:

- $R = k[[x_1,x_2,y_1,y_2,y_3]]$
- $\mathcal{X} = K^R(\underline{x})$: $0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R^2 \xrightarrow{(x_1 \;\; x_2)} R \longrightarrow 0$
- $\mathcal{Y} = K^R(\underline{y})$:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}} R^3 \xrightarrow{(y_1 \;\; y_2 \;\; y_3)} R \to 0$$

Thm (V '21, -):

The construction

$$(\mathcal{X} * \mathcal{Y})_i = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{i+1} & i \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & i = 0 \end{cases}, \qquad \partial_i^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{i+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & i \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & i = 1 \end{cases}$$

is a minimal resolution of $\dfrac{R}{\langle \underline{x}\, \underline{y} \rangle}$.

Example!

$$0 \to R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} R^9 \xrightarrow{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}} R^6 \xrightarrow{(x_1y_1 \;\; x_1y_2 \;\; x_1y_3 \;\; x_2y_1 \;\; x_2y_2 \;\; x_2y_3)} R \to 0$$

DGA?

$$(e_1 * f_2)(e_2 * f_2) = e_1 e_2 * y_2 f_2 = y_2 \; e_{12} * f_2$$

$$(e_1 * f_{23})(e_1 * f_3) = 0$$

$$(e_1 * f_{23})(e_2 * f_3) = 0$$

$$(e_1 * f_2)(e_2 * f_3) = y_2 \; e_{12} * f_3 - x_2 \; e_1 * f_{23}$$

Thm(-): The resolution $\mathcal{X} * \mathcal{Y}$ is a DGA where the product is given by

$$\begin{aligned}
(e_i * f_\Omega)(e_j * f_\Gamma) = & \mathbf{1}_{[\omega_1 \le \gamma_1 < \omega_2]} (-1)^{(|f_\Omega|-1)(|e_j|-1)} \left(e_i e_j * \mathrm{P}_1(f_\Omega) f_\Gamma\right) \\
& - \mathbf{1}_{[\omega_1 \le \gamma_1]} \mathbf{1}_{[|e_j|=1]} \left(e_i \partial^{\mathcal{X}}(e_j) * f_\Omega f_\Gamma\right) \\
& + \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]} (-1)^{(|f_\Omega|-1)|e_j|} \left(e_i e_j * f_\Omega \mathrm{P}_1(f_\Gamma)\right) \\
& - \mathbf{1}_{[\gamma_1 < \omega_1]} \mathbf{1}_{[|e_i|=1]} (-1)^{|f_\Omega|(|e_j|-1)} \left(\partial^{\mathcal{X}}(e_i) e_j * f_\Omega f_\Gamma\right).
\end{aligned}$$

Step 2!

$$\frac{k[[x_1,x_2,y_1,y_2,y_3]]}{(x_1^2, x_1x_2, x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3)} \cong \frac{k[[x_1,x_2]]}{(x_1^2, x_1x_2)} \times_k k[[\underline{y}]] \quad \text{over } k[[x_1,x_2,y_1,y_2,y_3]]$$

Strategy: set $I = (x_1^2, x_1x_2)$ and $\mathcal{S}$ the min res'l of R/I

$$0 \to I \cdot \frac{R}{\langle \underline{x}\,\underline{y} \rangle} \longrightarrow \frac{R}{\langle \underline{x}\,\underline{y} \rangle} \longrightarrow \frac{R}{\langle I, \underline{x}\,\underline{y} \rangle} \to 0$$

- $\dfrac{R}{\langle \underline{x}\,\underline{y} \rangle}$ resolved by $\mathcal{X} * \mathcal{Y}$
- $I \cdot \dfrac{R}{\langle \underline{x}\,\underline{y} \rangle} \cong I \otimes_R \dfrac{R}{\langle \underline{y} \rangle}$ resolved by $\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y})$
- Need $\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$

Build Φ from $\phi : \mathcal{S} \to \mathcal{X}$:

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\phi"] & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"] \arrow[d, "x_1"] & R^2 \arrow[r, "\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix}"] & R \arrow[r] \arrow[d, "1\cdot"] & 0 \\
\mathcal{X} & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"'] & R^2 \arrow[r, "\begin{pmatrix} x_1 & x_2 \end{pmatrix}"'] & R \arrow[r] & 0
\end{tikzcd}$$

Define $\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$

$$\Phi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha|+|\beta|} \phi(\alpha) * \beta & |\beta| > 0 \\ \partial_1^{\mathcal{S}}(\alpha) * \beta & |\beta| = 0, |\alpha| = 1 \\ 0 & |\beta| = 0, |\alpha| > 0 \end{cases}$$

Example:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\Sigma^{-1}(\mathcal{S}_{\geq 1}\otimes \mathcal{Y}) : \arrow[d, "\Phi"] & 0 \arrow[r] & R \arrow[r, "{\begin{pmatrix} -y_3 \\ y_2 \\ -y_1 \\ x_2 \\ -x_1 \end{pmatrix}}"] \arrow[d, "-x_1"] & R^5 \arrow[rr, "{\begin{pmatrix} y_2 & y_3 & 0 & 0 & 0 \\ -y_1 & 0 & y_3 & 0 & 0 \\ 0 & -y_1 & -y_2 & 0 & 0 \\ x_2 & 0 & 0 & y_3 & 0 \\ 0 & x_2 & 0 & -y_2 & 0 \\ 0 & 0 & x_2 & y_1 & 0 \\ -x_1 & 0 & 0 & 0 & y_3 \\ 0 & -x_1 & 0 & 0 & -y_2 \\ 0 & 0 & -x_1 & 0 & y_1 \end{pmatrix}}"] \arrow[d, "x_1 I_5"] & & R^9 \arrow[rr, "{\begin{pmatrix} -y_1 & -y_2 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & -y_2 & -y_3 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & y_1 & 0 & -y_3 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & y_1 & y_2 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & -y_2 & -y_3 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & y_1 & 0 & -y_3 \\ 0 & 0 & -x_1 & 0 & 0 & 0 & 0 & y_1 & y_2 \end{pmatrix}}"] \arrow[d, "-x_1 I_9"] & & R^7 \arrow[rr, "{\begin{pmatrix} x_2 & y_1 & y_2 & y_3 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & y_1 & y_2 & y_3 \end{pmatrix}}"] \arrow[d, "{\left(\begin{array}{c|c} 0 & \\ 0 & \\ 0 & x_1 I_6 \\ 0 & \\ 0 & \\ 0 & \end{array}\right)}"] & & R^2 \arrow[r] \arrow[d, "{\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}}"] & 0 \\
\mathcal{X} * \mathcal{Y} ; & 0 \arrow[r] & R \arrow[r, swap, "{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}}"] & R^5 \arrow[rr, swap, "{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}}"] & & R^9 \arrow[rr, swap, "{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}}"] & & R^6 \arrow[rr, swap, "{\begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_2y_1 & x_2y_2 & x_2y_3 \end{pmatrix}}"] & & R \arrow[r] & 0
\end{tikzcd}
$$

<u>Thm(-)</u>: Φ is a chain map and Cone(Φ) is a minimal resolution of $\frac{R}{\langle I, \Sigma y_\cdot \rangle}$. Moreover Cone$(\Phi)$ is a DG-module over $\mathcal{X} * \mathcal{Y}$ with

$$
\begin{aligned}
(e_\mu * f_\Omega)(\alpha \otimes f_\Gamma) &= -\mathbb{1}[|e_\mu| = 1](-1)^{(|\alpha|-1)|f_\Omega|} \partial^{\mathcal{X}}(e_\mu)\alpha \otimes f_\Omega f_\Gamma \\
&\quad + \mathbb{1}[\omega_1 \leq \gamma_1](-1)^{|\alpha|(|f_\Omega|-1)+|f_\Gamma|} e_\mu \phi(\alpha) * f_\Omega f_\Gamma .
\end{aligned}
$$

Thank You!