Minimal Free Resolutions of Fiber Products

Let $\Bbbk$ be a field.

$\underline{x} = x_1, \dots, x_m$

$\underline{y} = y_1, \dots, y_n$

$$\pi_S : S = \frac{\Bbbk[\underline{x}]}{I} \twoheadrightarrow \Bbbk$$

$$\pi_T : T = \frac{\Bbbk[\underline{y}]}{J} \twoheadrightarrow \Bbbk$$

$$S \times_{\Bbbk} T = \{ (s,t) \mid \pi_S(s) = \pi_T(t) \}$$

$$\cong \frac{\Bbbk[\underline{x}, \underline{y}]}{\langle I, \underline{x}\underline{y}, J \rangle}$$

$I \subseteq \langle \underline{x} \rangle$

$J \subseteq \langle \underline{y} \rangle$

Want to build a free resolution of $\dfrac{k[\underline{x},\underline{y}]}{\langle I, \underline{x}\,\underline{y}, J\rangle}$ over $R = k[\underline{x},\underline{y}]$.

$$F = \cdots \to F_2 \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \to 0$$

$$\ker \partial_i^F = \operatorname{Im} \partial_{i+1}^F \quad \text{for } i \geq 1$$

$$F_0 / \operatorname{Im} \partial_1^F \cong M$$

the module I want to resolve

$$k \cong \frac{k[x_1, x_2]}{\langle x_1, x_2\rangle} \quad \text{over } k[x_1, x_2]$$

$$X = 0 \to k[x_1,x_2] \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} k[x_1,x_2]^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} k[x_1,x_2] \to 0$$

$$k \cong \frac{k[y_1, y_2, y_3]}{\langle y_1, y_2, y_3 \rangle} \quad \text{over } k[\underline{y}]$$

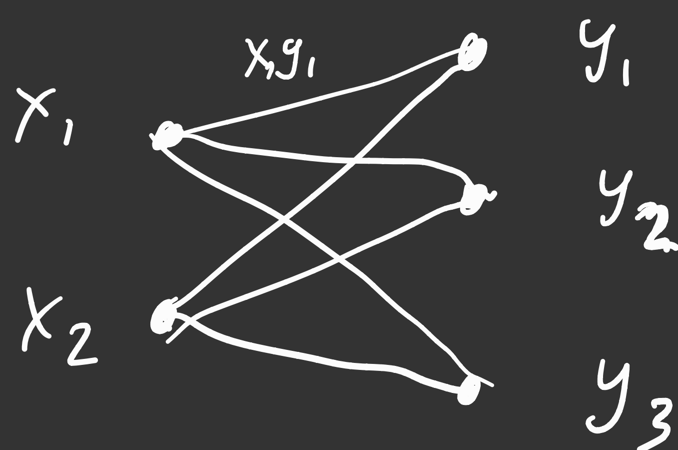
$$Y = \quad 0 \to k[\underline{y}] \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}} k[\underline{y}]^3 \xrightarrow{\begin{pmatrix} y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}} k[\underline{y}]^3 \xrightarrow{(y_1\ y_2\ y_3)} k[\underline{y}] \to 0$$

<u>Case 1</u>: $J = 0 \Rightarrow J$

Resolve $\dfrac{R}{\langle \underline{x}\underline{y} \rangle} = \dfrac{R}{I(K_{m,n})}$

This was resolved by Visscher '06

$$I(K_{2,3}) = \langle x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3 \rangle$$



$$0 \to R \xrightarrow[\begin{pmatrix} \partial_3^{y} \\ \partial_2^{x} \end{pmatrix}]{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} R^5 \xrightarrow[\begin{pmatrix} \partial_2^{y} & 0 & 0 \\ -x_2 I_3 & -\partial_3^{y} & 0 \\ x_1 I_3 & 0 & -\partial_3^{y} \end{pmatrix}]{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & +y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & +y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} R^9 \xrightarrow{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}} R^6 \longrightarrow R \to 0$$

$$\langle x_1 \; x_2 \rangle \langle y_1 \; y_2 \; y_3 \rangle = I(K_{2,3})$$

$$(\mathcal{X} * \mathcal{Y})_i = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_k \mathcal{Y}_{\geq 1})_{i+1} & i \geq 1 \\ \mathcal{X}_0 \otimes_k \mathcal{Y}_0 & i = 0 \end{cases}$$

$$(\cong R)$$

$$\partial_i^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{i+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & i \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & i = 1 \end{cases}$$

<u>Thm</u>: The construction $\mathcal{X} * \mathcal{Y}$ is a minimal resolution of $k[\underline{x}] \times_k k[\underline{y}]$.

<u>Recall</u>: $k[\underline{x}] \times_k k[\underline{y}] \cong \dfrac{R}{\langle \underline{x}\,\underline{y} \rangle}$

<u>Case 2</u>: $\frac{k[\underline{x}]}{I} \times_k k[\underline{y}]$, $I \subseteq \langle \underline{x} \rangle^2$

$$\frac{k[x]}{\langle x^3 \rangle} \times_k k[y_1, y_2] \cong \frac{k[x, y_1, y_2]}{\langle x^3, xy_1, xy_2 \rangle}$$

$$X * Y = 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} R^2 \xrightarrow{(xy_1,\ xy_2)} R \longrightarrow 0$$

Minimal resolution of $\frac{k[x]}{\langle x^3 \rangle}$ over $k[x]$.

$$\begin{tikzcd}
\mathcal{S} = \arrow[d, "\phi"] & 0 \arrow[r] & k[x] \arrow[r, "x^3"] \arrow[d, "x^2"] \arrow[dr, phantom, "\circlearrowright"] & k[x] \arrow[r] \arrow[d, "1"] & 0 \\
X = & 0 \arrow[r] & k[x] \arrow[r, "x"'] & k[x] \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
\Sigma^{-1}(\mathcal{S}_{Z_1} \otimes Y) = 0 \arrow[r] \arrow[d, "\Phi"'] & R \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "x^2"] & R^2 \arrow[r, "(y_1\ y_2)"] \arrow[d, "\begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}"] & R \arrow[r] \arrow[d, "x^3"] & 0 \\
X * Y = 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"'] & R^2 \arrow[r, "(xy_1\ xy_2)"'] & R \arrow[r] & 0
\end{tikzcd}
$$

$\mathrm{Cone}(\Phi) =$

$$
0 \longrightarrow \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \xrightarrow{\begin{pmatrix} x^2 \\ y_2 \\ -y_1 \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R^2 \end{matrix} \xrightarrow{\begin{pmatrix} -y_2 & x^2 & 0 \\ y_1 & 0 & x^2 \\ 0 & -y_1 & -y_2 \end{pmatrix}} \begin{matrix} R^2 \\ \oplus \\ R \end{matrix} \xrightarrow{(xy_1\ xy_2\ x^3)} \begin{matrix} R \\ \oplus \\ 0 \end{matrix} \to 0
$$

Minimal resolution of $\dfrac{k[x, y_1, y_2]}{\langle x^3, xy_1, xy_2 \rangle}$

In general, given $\phi : \mathcal{S} \longrightarrow X$

we define Φ as

$$\Phi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha|+|\beta|}\phi(\alpha) * \beta & |\beta|>0 \\ \partial_1^{\mathcal{S}}(\alpha) * \beta & |\beta|=0,\ |\alpha|=1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u>: If $\mathcal{S}$ is a minimal res'l of $\frac{k[\underline{x}]}{I}$ with $I \subseteq \langle \underline{x} \rangle^2$, then $\mathrm{Cone}(\Phi)$ minimally resolves $\frac{k[\underline{x}]}{I} \times_k k[\underline{y}]$.

What if $I = 0$ and $J \subseteq \langle \underline{y} \rangle^2$?

Suppose $\mathcal{Z}$ is a min res'l of $\frac{k[\underline{y}]}{J}$

$\psi : \mathcal{Z} \longrightarrow \mathcal{Y}$ we build Ψ

$$\Psi : \Sigma^{-1}(\mathcal{X} \otimes_k \mathcal{Z}_{\geq 1}) \longrightarrow \mathcal{X} * \mathcal{Y}$$

where

$$\Psi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha|+|\beta|-1} \alpha * \psi(\beta) & |\alpha| > 0 \\ \alpha * \partial_1^{\mathcal{Z}}(\beta) & |\alpha| = 0, |\beta| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thm:

If $\mathcal{Z}$ is a min res'l of $\frac{k[\underline{y}]}{J}$ with $J \subseteq \langle \underline{y} \rangle^2$, then $\operatorname{Cone}(\Psi)$ is a min res'l of $k[\underline{x}] \times_k \frac{k[\underline{y}]}{J}$.

Thm: Under the same conditions, we have $\operatorname{Cone}(\Phi\ \Psi)$ is the min resolution of $\frac{k[\underline{x}]}{I} \times_k \frac{k[\underline{y}]}{J}$.

$(R, \mathfrak{m}, k)$

Minimal free resolution F satisfies

$$\operatorname{Im} \partial^F_{i+1} \subseteq \mathfrak{m} F_i$$

$$\operatorname{Im} \partial^F \subseteq \mathfrak{m} F$$