Classifying Betti Numbers of Fiber Products

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April 16, 2023

Fiber Products

Let S, T, and W be local rings with ring homomorphisms $\pi_S : S \to W \leftarrow T : \pi_T$. The **Fiber Product** of S and T over W is the ring

$$S \times_W T := \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}.$$

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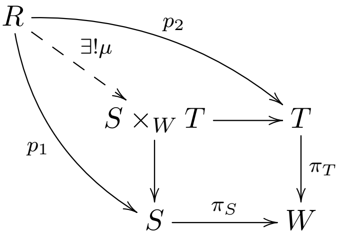
Universal Mapping Property:



Why study Fiber Products?

1. **Avramov, Foxby, Herzog '94:** Any two Cohen Factorizations have a common deformation.
2. **Christensen, Striuli, Veliche '10:** Studied the depth $S \times_k T$.
3. **Nasseh, Sather-Wagstaff '17:** $R = S \times_k T$ satisfies the Auslander-Reiten Conjecture.
4. **Celikbas, Celikbas, Ciuperca, Endo, Goto, Isobe, Matsuoka:** Study Arf rings arising from fiber products $S \times_W T$.

Resolutions of Fiber Products:

Theorem (G'22)

Let (R, k) *be a local ring with ideals* $\mathcal{I}' \subseteq \mathcal{I}$ *and* $\mathcal{J}' \subseteq \mathcal{J}$ *with certain* Tor*-vanishing conditions. Set* $S = \frac{R}{\mathcal{I}'+\mathcal{J}}$, $T = \frac{R}{\mathcal{I}+\mathcal{J}'}$, *and* $W = \frac{R}{\mathcal{I}+\mathcal{J}}$. *One can construct a free resolution of* $S \times_W T$ *over* R *from free resolutions of* $R/\mathcal{I}$, $R/\mathcal{I}'$, $R/\mathcal{J}$, *and* $R/\mathcal{J}'$.

If $\operatorname{Tor}_i^R(\frac{R}{\mathcal{I}'}, k) \to \operatorname{Tor}_i^R(\frac{R}{\mathcal{I}}, k)$ *and* $\operatorname{Tor}_i^R(\frac{R}{\mathcal{J}'}, k) \to \operatorname{Tor}_i^R(\frac{R}{\mathcal{J}}, k)$ *are zero maps for all* $i \geq 1$, *then this construction is minimal.*

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Corollary (G'22)

For $\ell \geq 1$, *we have*

$$\beta_\ell^R(S \times_W T) = \beta_\ell^R\left(\frac{R}{\mathcal{I}\mathcal{J}}\right) + \sum_{t=1}^{\ell}\left(\beta_t^R\left(\frac{R}{\mathcal{I}'}\right)\beta_{\ell-t}^R\left(\frac{R}{\mathcal{J}}\right) + \beta_{\ell-t}^R\left(\frac{R}{\mathcal{I}}\right)\beta_t^R\left(\frac{R}{\mathcal{J}'}\right)\right).$$

Resolutions of Fiber Products Revised:

Theorem (G'22)

Let $R = k[\![\underline{x}, \underline{y}]\!]$ be a local ring with ideals $\mathcal{I}' \subseteq (\underline{x})$ and $\mathcal{J}' \subseteq (\underline{y})$. Set $S = \frac{k[\![\underline{x}]\!]}{\mathcal{I}'}$, $T = \frac{k[\![\underline{y}]\!]}{\mathcal{J}'}$, and $W = k$. One can construct a free resolution of $S \times_W T$ over R from those of $R/\mathcal{I}'$ and $R/\mathcal{J}'$.

If $\mathcal{I}' \subseteq (\underline{x})^2$ and $\mathcal{J}' \subseteq (\underline{y})^2$, then this construction is minimal.

Resolutions of Fiber Products Revised:

Theorem (G'22)

Let $R = k[\![\underline{x}, \underline{y}]\!]$ *be a local ring with ideals* $\mathcal{I}' \subseteq (\underline{x})$ *and* $\mathcal{J}' \subseteq (\underline{y})$*. Set* $S = \frac{k[\![\underline{x}]\!]}{\mathcal{I}'}$, $T = \frac{k[\![\underline{y}]\!]}{\mathcal{J}'}$*, and* $W = k$*. One can construct a free resolution of* $S \times_W T$ *over* R *from those of* $R/\mathcal{I}'$ *and* $R/\mathcal{J}'$*.*

If $\mathcal{I}' \subseteq (\underline{x})^2$ *and* $\mathcal{J}' \subseteq (\underline{y})^2$*, then this construction is minimal.*

Corollary (G'22)

If $\underline{x} = x_1, \ldots, x_n$ *and* $\underline{y} = y_1, \ldots, y_{n'}$*, then for* $\ell \geq 1$*, we have*

$$\beta_\ell^R(S \times_W T) = \binom{n+n'}{\ell+1} - \binom{n}{\ell+1} - \binom{n'}{\ell+1} + \sum_{t=1}^{\ell} \left(\beta_t^R\left(\frac{R}{\mathcal{I}'}\right)\binom{n'}{\ell-t} + \binom{n}{\ell-t}\beta_t^R\left(\frac{R}{\mathcal{J}'}\right) \right).$$

Question:

1. **Today:** What if $\mathcal{I}' \not\subseteq (\underline{x})^2$?

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1. **Today:** What if $\mathcal{I}' \not\subseteq (\underline{x})^2$?
2. **Future:** What if for some $i \geq 1$ we have $\operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}'}, k\right) \to \operatorname{Tor}_i^R\left(\frac{R}{\mathcal{I}}, k\right)$ is not the zero map?

Example Set-up:

- $R = k[\![x_1, x_2, y_1, y_2]\!]$
- $\mathcal{I}' = \left(x_1^3 + x_2^2\right) \subset k[\![x_1, x_2]\!]$
- $S = \frac{k[\![x_1, x_2]\!]}{\mathcal{I}'} \cong \frac{R}{\mathcal{I}' + (y_1, y_2)}$
- $\mathcal{J}' = \left(y_1^3 + y_2^2\right) \subseteq k[\![y_1, y_2]\!]$
- $T = \frac{k[\![y_1, y_2]\!]}{\mathcal{J}'} \cong \frac{R}{(x_1, x_2) + \mathcal{J}'}$

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Note:

$$S \times_k T \cong \frac{R}{\mathcal{I}' + (x_1y_1, x_1y_2, x_2y_1, x_2y_2) + \mathcal{J}'}$$

Example Resolutions:

1. $\mathcal{S} = 0 \longrightarrow R \xrightarrow{(x_1^3+x_2^2)} R \longrightarrow 0$

2. $\mathcal{X} = 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} R \longrightarrow 0$

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3. $\mathcal{T} = 0 \longrightarrow R \xrightarrow{(y_1^3+y_2^2)} R \longrightarrow 0$

4. $\mathcal{Y} = 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} R \longrightarrow 0$

Step 1:

The **star product** of $\mathcal{X}$ and $\mathcal{Y}$ over R, denoted $\mathcal{X} *_R \mathcal{Y}$, is the chain complex given by

$$(\mathcal{X} *_R \mathcal{Y})_n = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{n+1} & n \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & n = 0 \\ 0 & n < 0 \end{cases}$$

$$\partial_n^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{n+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & n \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & n = 1 \\ 0 & n \leq 0 \end{cases}.$$

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$$0 \to R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x_1y_1 & x_1y_2 & x_2y_1 & x_2y_2 \end{pmatrix}} R \to 0$$

Step 2:

$$\begin{array}{ccccccccc}
\mathcal{S}=0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{(x_1^3+x_2^2)} & R & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix}} & & \downarrow{\scriptstyle 1} & & \\
\mathcal{X}=0 & \longrightarrow & R & \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}]{} & R^2 & \xrightarrow[(\,x_1\ \ x_2\,)]{} & R & \longrightarrow & 0
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\mathcal{X} = 0 & \longrightarrow & R & \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}]{} & R^2 & \xrightarrow[\begin{pmatrix} x_1 & x_2 \end{pmatrix}]{} & R & \longrightarrow & 0
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$$

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\begin{array}{lcccccccc}
\Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes_R \mathcal{Y}) = 0 \to 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} & R \to 0 \\
\downarrow \Phi & & \downarrow \begin{pmatrix} -x_1^2 \\ -x_2 \\ 0 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} x_1^2 & 0 \\ 0 & x_1^2 \\ x_2 & 0 \\ 0 & x_2 \end{pmatrix} & & \downarrow (x_1^3+x_2^2) \\
\mathcal{X} *_R \mathcal{Y} \qquad = 0 \to R & \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}]{} & R^4 & \xrightarrow[(\underline{xy})]{} & R \to 0
\end{array}
$$

Step 3:

$$
\begin{array}{ccccccccc}
\mathcal{T}=0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{(y_1^3+y_2^2)} & R & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \begin{pmatrix} y_1^2 \\ y_2 \end{pmatrix}} & & \downarrow{\scriptstyle 1} & & \\
\mathcal{Y}=0 & \longrightarrow & R & \xrightarrow[\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}]{} & R^2 & \xrightarrow[\begin{pmatrix} y_1 & y_2 \end{pmatrix}]{} & R & \longrightarrow & 0
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\mathcal{X} *_R \mathcal{Y} \qquad = 0 \to R & \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}]{} & R^4 & \xrightarrow[(\underline{xy})]{} & R \to 0
\end{array}
$$

Final Step:

Desired resolution given by the mapping cone of the below.

$$
\begin{array}{ccccccc}
0 \to 0 \longrightarrow & R^2 & \xrightarrow{\begin{pmatrix} -y_2 & 0 \\ y_1 & 0 \\ 0 & x_2 \\ 0 & -x_1 \end{pmatrix}} & R^4 & \xrightarrow{\begin{pmatrix} y_1 & y_2 & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 \end{pmatrix}} & R^2 & \to 0 \\
& \Big\downarrow {\scriptstyle \begin{pmatrix} -x_1^2 & 0 \\ -x_2 & 0 \\ 0 & y_1^2 \\ 0 & y_2 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} x_1^2 & 0 & -y_1^2 & 0 \\ 0 & x_1^2 & -y_2 & 0 \\ x_2 & 0 & 0 & -y_1^2 \\ 0 & x_2 & 0 & -y_2 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} x_1^3+x_2^2 & 0 \\ 0 & y_1^3+y_2^2 \end{pmatrix}} & \\
0 \to R \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} x_1y_1 & x_1y_2 & x_2y_1 & x_2y_2 \end{pmatrix}]{} & R & \to 0
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Breaking the Example

What if we replace $\mathcal{I}' = (x_1^3 + x_2^2)$ with $\mathcal{I}' = (x_1 + x_1^3 + x_2^2)$?

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\mathcal{X} = 0 & \longrightarrow & R & \xrightarrow[\binom{-x_2}{x_1}]{} & R^2 & \xrightarrow[(\,x_1\ x_2\,)]{} & R & \longrightarrow & 0
\end{array}
$$

Observe: $\operatorname{Tor}_1^R(\phi, k) \neq 0$

Breaking the Example Cont.

$$
\begin{array}{ccccccccc}
0 \to 0 & \longrightarrow & R^2 & \xrightarrow{\begin{pmatrix} -y_2 & 0 \\ y_1 & 0 \\ 0 & x_2 \\ 0 & -x_1 \end{pmatrix}} & R^4 & \xrightarrow{\begin{pmatrix} y_1 & y_2 & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 \end{pmatrix}} & R^2 & \longrightarrow & 0 \\
& & \Big\downarrow \begin{pmatrix} -1-x_1^2 & 0 \\ -x_2 & 0 \\ 0 & 1+y_1^2 \\ 0 & y_2 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1+x_1^2 & 0 & -1-y_1^2 & 0 \\ 0 & 1+x_1^2 & -y_2 & 0 \\ x_2 & 0 & 0 & -1-y_1^2 \\ 0 & x_2 & 0 & -y_2 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} x_1+x_1^3+x_2^2 & 0 \\ 0 & y_1+y_1^3+y_2^2 \end{pmatrix} & & \\
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 & & \Big\downarrow {\scriptstyle \begin{pmatrix} -1-x_1^2 & 0 \\ -x_2 & 0 \\ 0 & 1+y_1^2 \\ 0 & y_2 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1+x_1^2 & 0 & -1-y_1^2 & 0 \\ 0 & 1+x_1^2 & -y_2 & 0 \\ x_2 & 0 & 0 & -1-y_1^2 \\ 0 & x_2 & 0 & -y_2 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} x_1+x_1^3+x_2^2 & 0 \\ 0 & y_1+y_1^3+y_2^2 \end{pmatrix}} & & \\
0 \to R & \xrightarrow[\begin{pmatrix} -x_2 \\ x_1 \\ -y_2 \\ y_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} y_2 & 0 & -x_2 & 0 \\ -y_1 & 0 & 0 & -x_2 \\ 0 & y_2 & x_1 & 0 \\ 0 & -y_1 & 0 & x_1 \end{pmatrix}]{} & R^4 & \xrightarrow[\begin{pmatrix} x_1y_1 & x_1y_2 & x_2y_1 & x_2y_2 \end{pmatrix}]{} & R & \longrightarrow & 0
\end{array}
$$

Observe:

$$
S \times_k T \cong \frac{R}{\mathcal{I}' + (x_1y_1, x_1y_2, x_2y_1, x_2y_2) + \mathcal{J}'} \cong \frac{R}{\mathcal{I}' + (x_2y_2) + \mathcal{J}'}
$$

Generalization

Lemma

Let $I = (g_1, \ldots, g_m) \subseteq k[\![\underline{x}]\!]$ *with* $p = \dim_k \left(\frac{I + (\underline{x})^2}{(\underline{x})^2} \right)$*, then a change of variable allows allows one to set* $g_i = x_i$ *for* $1 \leq i \leq p$*.*

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- Set $I = (x_1, \ldots, x_p) + I_0$ with $I_0 \subseteq (x_{p+1}, \ldots, x_n)^2$.
- Set $J = (y_1, \ldots, y_q) + J_0$ with $J_0 \subseteq (y_{q+1}, \ldots, y_{n'})^2$.

Generalization

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Let $I = (g_1, \ldots, g_m) \subseteq k[\![\underline{x}]\!]$ *with* $p = \dim_k \left(\frac{I + (\underline{x})^2}{(\underline{x})^2} \right)$*, then a change of variable allows allows one to set* $g_i = x_i$ *for* $1 \leq i \leq p$*.*

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- Set $J = (y_1, \ldots, y_q) + J_0$ with $J_0 \subseteq (y_{q+1}, \ldots, y_{n'})^2$.

Theorem

Set $R' = k[\![x_{p+1}, \ldots, x_n, y_{q+1}, \ldots, y_{n'}]\!]$*. The fiber product* $S \times_k T$ *is isomorphic (as* R*-algebras) to the following.*

$$\left(\frac{k[\![x_{p+1}, \ldots, x_n]\!]}{I_0} \times_k \frac{k[\![y_{q+1}, \ldots, y_{n'}]\!]}{J_0} \right) \otimes_{R'} \frac{R}{(x_1, \ldots, x_p, y_1, \ldots, y_q)}$$

Classifying Betti Numbers

Theorem

The Betti numbers of the fiber product $F = \frac{k[\![\underline{x}]\!]}{I} \times_k \frac{k[\![\underline{y}]\!]}{J}$ *over* $R = k[\![\underline{x}, \underline{y}]\!]$ *are given by* $\beta_0^R(F) = 1$ *and, for* $t \geq 1$*,*

$$\beta_t^R(F) = \binom{n+n'}{t+1} - \binom{n'+p+1}{t+1} - \binom{n+q+1}{t+1} + \binom{p+q+1}{t+1} + \sum_{w+z=t} \left(\beta_w^R\left(\frac{R}{I}\right)\binom{n'}{z} + \binom{n}{z}\beta_w^R\left(\frac{R}{I'}\right)\right).$$

Thank you!