DG Structures for Products of Ideals

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DG Algebras

Let R be a local ring. A commutative differential graded (DG) R-algebra is an R-complex with a binary operation $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following for all $a, b, c \in \mathcal{X}$:

- Associative: $(ab)c = a(bc)$.
- Distributive: $(a + b)c = ac + bc$ if $|a| = |b|$.
- Unital: there exists an element $1_{\mathcal{X}} \in \mathcal{X}_0$ such that $1_{\mathcal{X}} a = a$ for all $a \in \mathcal{X}$.
- Graded commutative: $ab = (-1)^{|a||b|} ba \in \mathcal{X}_{|a|+|b|}$ and $a^2 = 0$ if $|a|$ is odd.
- Leibniz rule: $\partial_{|a|+|b|}(ab) = \partial_{|a|}(a)b + (-1)^{|a|} a\partial_{|b|}(b)$.

If, in addition, $\mathcal{X}$ is a free resolution, we say it as a DGA resolution.

DG Algebra Examples

- Short free resolutions of R/I
- Koszul Complex $K^R(r_1, \ldots, r_n)$:

$$e_I e_J := \begin{cases} \operatorname{sign}(I, J)\, e_{I \cup J} & I \cap J = \emptyset \\ 0 & I \cap J \neq \emptyset \end{cases}$$

- Taylor Resolution $T^R(m_1, \ldots, m_n)$:

$$e_I e_J := \begin{cases} \operatorname{sign}(I, J)\, \frac{m_I m_J}{m_{I \cup J}} e_{I \cup J} & I \cap J = \emptyset \\ 0 & I \cap J \neq \emptyset \end{cases}$$

- Tensor Products of DGAs:

$$(a \otimes b)(c \otimes d) = (-1)^{|b| \cdot |c|} (ac \otimes bd)$$

DG Algebra Examples:

Let $R = k[\![x_1, x_2, y_1, y_2, y_3]\!]$ and consider $R/(\underline{x})$ and $R/(\underline{y})$.

$$\mathcal{X} := 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} R \longrightarrow 0$$

Basis: $\{1, e_1, e_2, e_{12}\}$
Product: $e_1 e_2 = e_{12} = -e_2 e_1$

$$\mathcal{Y} := 0 \longrightarrow R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}} R \longrightarrow 0$$

Basis: $\{1, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}, f_{123}\}$
Product: $f_1 f_{23} = f_{123} = f_{23} f_1$ and $f_2 f_{13} = -f_{123} = f_{13} f_2$

Minimality vs Product

Fact: Every R-algebra has a DG algebra resolution over R but there exist some R-algebras with minimal resolutions that **cannot** have a DG structure.

Theorem (Avramov '81)

Let $(R, \mathfrak{m}, k)$ denote a local ring and $f : R \to S$ a morphism of rings. Suppose the minimal free resolution of S over R, denoted F, is a DGA. If

$$o^f(M) := \ker\left(\frac{\mathrm{Tor}^R(M,k)}{\mathrm{Tor}^S_+(M,k)\cdot \mathrm{Tor}^R(M,k)} \to \mathrm{Tor}^S(M,k)\right) \neq 0,$$

then no DG F-module structure exists on the minimal free resolution of M over R.

Motivation and Notation:

Motivation: Given a local ring R and ideals $\mathcal{I}, \mathcal{J} \subseteq R$, use the minimal DG algebra resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ to construct a minimal DG algebra resolution of $R/\mathcal{I}\mathcal{J}$.

Running Example:

- Let k be field and set $R = k[\![\underline{x}, \underline{y}]\!]$.
- Set $\mathcal{I} = (\underline{x}) = (x_1, x_2)$ and $\mathcal{J} = (\underline{y}) = (y_1, y_2, y_3)$.
- Observe $\mathcal{I}\mathcal{J} = (x_i y_j : 1 \leq i \leq 2, 1 \leq j \leq 3) =: (\underline{xy})$.

Observation: The ideal $(\underline{xy})$ is the edge ideal of $K_{2,3}$.

Star Product Construction

Construction

Let $\mathcal{X}$ and $\mathcal{Y}$ be complexes of free R-modules. The **star product** *of $\mathcal{X}$ and $\mathcal{Y}$ over R, denoted $\mathcal{X} *_R \mathcal{Y}$, is the chain complex given by*

$$(\mathcal{X} *_R \mathcal{Y})_n = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_R \mathcal{Y}_{\geq 1})_{n+1} & n \geq 1 \\ \mathcal{X}_0 \otimes_R \mathcal{Y}_0 & n = 0 \\ 0 & n < 0 \end{cases}$$

$$\partial_n^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{n+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & n \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & n = 1 \\ 0 & n \leq 0 \end{cases}.$$

Star Product Resolution

Theorem (Vandebogert'22, G'22)

Let $\mathcal{X}$ and $\mathcal{Y}$ be free resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ over R, respectively. If $\mathcal{I}, \mathcal{J} \subseteq R$ are Tor-independent ideals, then the star product $\mathcal{X} *_R \mathcal{Y}$ is a free resolution of $R/\mathcal{I}\mathcal{J}$ over R. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are minimal, then $\mathcal{X} *_R \mathcal{Y}$ is also minimal.

Running Example:

$$(x_1, x_2) \cap (y_1, y_2, y_3) = (\underline{xy}) \Longrightarrow \mathrm{Tor}_i^R\left(\frac{R}{(\underline{x})}, \frac{R}{(\underline{y})}\right) = 0, i \geq 1$$

$$\mathcal{X} * \mathcal{Y} = 0 \to R \to R^5 \to R^9 \to R^6 \to R \to 0$$

Example Continued

$$\partial_2 = \begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}$$

Basis of $(\mathcal{X} * \mathcal{Y})_2$:

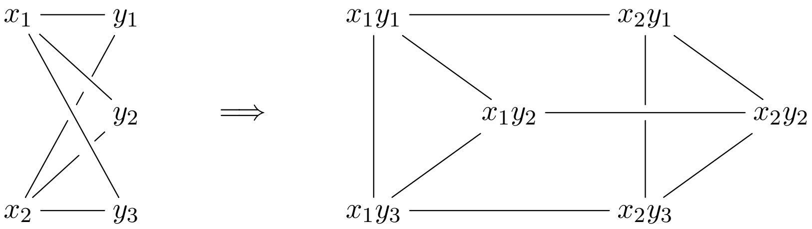
$$\{e_{12}*f_1, e_{12}*f_2, e_{12}*f_3, e_1*f_{12}, e_1*f_{13}, e_1*f_{23}, e_2*f_{12}, e_2*f_{13}, e_2*f_{23}\}$$

Basis of $(\mathcal{X} * \mathcal{Y})_1$:

$$\{e_1 * f_1, e_1 * f_2, e_1 * f_3, e_2 * f_1, e_2 * f_2, e_2 * f_3\}$$

Complete Bipartite Graphs

Original Construction: Done by Visscher '06 using Bayer, Peeva, Sturmfels '98



$$\mathcal{X} * \mathcal{Y} = 0 \to R \to R^5 \to R^9 \to R^6 \to R \to 0$$

DGA Reasonable

Theorem (Vandebogert '22)

Suppose $\mathcal{I}, \mathcal{J} \subseteq R$ are Tor-independent ideals. If the minimal free resolutions of $R/\mathcal{I}$ and $R/\mathcal{J}$ both have a DGA structure, then the Avramov Obstruction vanishes for $R/\mathcal{I}\mathcal{J}$.

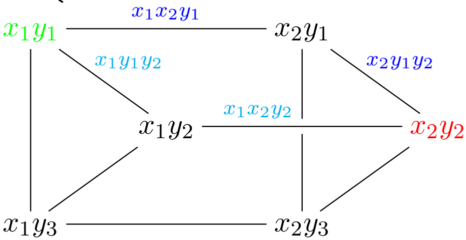
Theorem (Sköldberg)

For a cointerval graph G, the minimal free resolution over R of the edge ideal I_G has a DGA structure.

Vandebogert Product

Let $e_i, e_I \in \mathcal{X}$ and $f_j, f_J \in \mathcal{Y}$, then

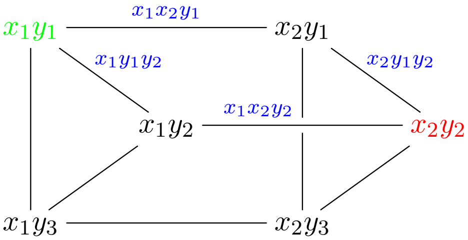
$$(e_i * f_j)(e_I * f_J) = \begin{cases} (-1)^{|I|}\partial(e_i)e_I * f_j f_J & |J| > 1 \\ (-1)^{|I|}\partial(e_i)e_I * f_j f_J + e_i e_I * f_j \partial(f_J) & |J| = 1 \end{cases}$$



$$(e_1 * f_1)(e_2 * f_2) = y_2(e_1 e_2 * f_1) - x_1(e_2 * f_1 f_2)$$
$$(e_2 * f_2)(e_1 * f_1) = y_1(e_2 e_1 * f_2) - x_2(e_1 * f_2 f_1)$$

Averaging Product

Idea: Average over pathways!



$$(e_1 * f_1)(e_2 * f_2) = \frac{1}{2}\left(y_2(e_1e_2 * f_1) - x_1(e_2 * f_1f_2)\right) - \frac{1}{2}\left(y_1(e_2e_1 * f_2) - x_2(e_1 * f_2f_1)\right)$$

General formula is not associative!

Sköldberg Product

Lemma (Sköldberg)

Let $e_i * f_j, e_a * f_b \in (\mathcal{X} * \mathcal{Y})_1$, *then*

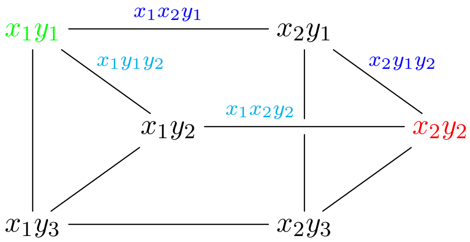
$$(e_i * f_j)(e_a * f_b) = \begin{cases} x_i(e_a * f_b f_j) + y_b(e_i e_a * f_j) & j > b, i < a \\ x_i(e_a * f_b f_j) & j > b, i = a \\ x_i(e_a * f_b f_j) - y_b(e_a e_i * f_j) & j > b, i > a \\ y_j(e_i e_a * f_b) & j = b, i < a \\ 0 & j = b, i = a \\ -y_b(e_a e_i * f_j) & j = b, i > a \\ y_j(e_i e_a * f_b) - x_a(e_i * f_j f_b) & j < b, i < a \\ -x_a(e_i * f_j f_b) & j < b, i = a \\ -x_a(e_i * f_j f_b) - y_j(e_a e_i * f_b) & j < b, i > a \end{cases}$$

Sköldberg Product Revised

Lemma (Sköldberg)

Let $e_i * f_j, e_a * f_b \in (\mathcal{X} * \mathcal{Y})_1$*, then*

$$(e_i * f_j)(e_a * f_b) = \begin{cases} e_i e_a * f_j \partial(f_b) - \partial(e_i) e_a * f_j f_b & j \geq b \\ e_i e_a * \partial(f_j) f_b - e_i \partial(e_a) * f_j f_b & j \leq b \end{cases}$$



$$(e_1 * f_1)(e_2 * f_2) = y_1(e_1e_2 * f_2) - x_2(e_1 * f_1 f_2) = -(e_2 * f_2)(e_1 * f_1)$$

The DGA Structure

Original attempt:

$$(e_I * f_\Omega)(e_J * f_\Gamma) = \begin{cases} \pm e_I e_J * f_\Omega \partial(f_\Gamma) \pm \partial(e_I) e_J * f_\Omega f_\Gamma & \omega_1 \geq \gamma_1 \\ \pm e_I e_J * \partial(f_\Omega) f_\Gamma \pm e_I \partial(e_J) * f_\Omega f_\Gamma & \omega_1 \leq \gamma_1 \end{cases}$$

Theorem

Suppose $\mathcal{X}$ is a minimal resolution of $R/(\underline{x})$ and $\mathcal{Y}$ is a minimal resolution of $R/(\underline{y})$. The resolution $\mathcal{X} *_R \mathcal{Y}$ is a minimal DGA resolution of $R/\langle \underline{xy} \rangle$ where the multiplication is given by

$$\begin{aligned} (e_\alpha * f_\Omega)(e_\beta * f_\Gamma) = & \mathbf{1}_{[\omega_1 \leq \gamma_1 < \omega_2]} (-1)^{(u-1)(b-1)} e_\alpha e_\beta * \mathrm{P}_1 (f_\Omega) f_\Gamma \\ & - \mathbf{1}_{[\omega_1 < \gamma_1]} \mathbf{1}_{[b=1]} e_\alpha \partial (e_\beta) * f_\Omega f_\Gamma \\ & + \mathbf{1}_{[\gamma_1 < \omega_1 < \gamma_2]} (-1)^{(u-1)b} e_\alpha e_\beta * f_\Omega \mathrm{P}_1 (f_\Gamma) \\ & - \mathbf{1}_{[\gamma_1 < \omega_1]} \mathbf{1}_{[a=1]} (-1)^{u(b-1)} \partial (e_\alpha) e_\beta * f_\Omega f_\Gamma . \end{aligned}$$

Generalization

Known Generalizations:

1. Do not need $\mathcal{X}$ and $\mathcal{Y}$ to resolutions or minimal.
2. Can replace $\mathcal{X}$ with any DGA.
3. The complex $\mathcal{Y}$ must be supported on a simplex.
4. The complex $\mathcal{Y}$ must have *simplicial multiplication*.

Desired Generalization: Drop all simplicial requirements!

Thank you!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Haydee Lindo - Harvey Mudd College, U.S.
Linquan Ma - Purdue University, U.S.
Sarasij Maitra - University of Utah, U.S.
Takumi Murayama - Purdue University, U.S.
Rebecca R.G. - George Mason University, U.S.
Liana Şega - University of Missouri-Kansas City, U.S.
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