DG-Structures For Fiber Products

Let S, T be rings with common residue field $\Bbbk$.

$\pi_S : S \longrightarrow \Bbbk$

$\pi_T : T \longrightarrow \Bbbk$

$$S \times_{\Bbbk} T = \{ (s,t) \in S \times T \mid \pi_S(s) = \pi_T(t) \}.$$

<u>Thm (Nasseh, Sather-Wagstaff '17)</u>

Let M and N be f.g. modules over the fiber product $F = S \times_{\Bbbk} T$ (S, T are local).

If $\mathrm{Tor}_i^F(M,N) = 0 = \mathrm{Tor}_{i+1}^F(M,N)$ for some $i \geq 5$, then $\mathrm{pd}_F M \leq 1$ or $\mathrm{pd}_F N \leq 1$.

Graded case: $S = \dfrac{\Bbbk[\underline{x}]}{I}$, $T = \dfrac{\Bbbk[\underline{y}]}{J}$

$$\Rightarrow \quad S \otimes_{\Bbbk} T \cong \frac{\Bbbk[\underline{x}, \underline{y}]}{\langle I, \underline{x}\underline{y}, J \rangle} \quad \text{where } \underline{x}\underline{y} = \{x_i y_j\}$$

- $\mathcal{S}$ is the minimal free resolution of $\dfrac{\Bbbk[\underline{x}]}{I}$ over $\Bbbk[\underline{x}]$, $I \subseteq \langle \underline{x} \rangle^2$
- $\mathcal{T}$ is the minimal free resolution of $\dfrac{\Bbbk[\underline{y}]}{J}$ over $\Bbbk[\underline{y}]$, $J \subseteq \langle \underline{y} \rangle^2$
- $\mathcal{X}$ is the Koszul complex on $\underline{x}$ over $\Bbbk[\underline{x}]$
- $\mathcal{Y}$ is the Koszul complex on $\underline{y}$ over $\Bbbk[\underline{y}]$

Case 1: $I = 0 = J$

- Visscher '06 gave an explicit construction of a minimal free resolution of

$$\frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle} \cong k[\underline{x}] \times_k k[\underline{y}].$$

- Unpublished paper claims these resolutions are DGAs. Paper has linearity issues.

Example: We can make the minimal resolution of

$$\frac{k[x_1, x_2, y_1, y_2, y_3]}{\langle x_1y_1 \;\; x_1y_2 \;\; x_1y_3 \;\; x_2y_1 \;\; x_2y_2 \;\; x_2y_3\rangle}$$

into a DGA.

$$0 \to R \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} R^9 \xrightarrow{\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}} R^6 \xrightarrow{\begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_2y_1 & x_2y_2 & x_2y_3 \end{pmatrix}} R \to 0$$

$$e_{x_1y_2} \cdot e_{x_2y_3} = y_2\, e_{x_1x_2y_3} - x_2\, e_{x_1y_2y_3}$$

$$e_{x_1y_2}\, e_{x_2y_2} = y_2\, e_{x_1x_2y_2}$$

$$e_{x_1y_2y_3}\, e_{x_2y_3} = 0$$

Construction:

$$(\mathcal{X} * \mathcal{Y})_i = \begin{cases} (\mathcal{X}_{\geq 1} \otimes_{\Bbbk} \mathcal{Y}_{\geq 1})_{i+1} & i \geq 1 \\ \mathcal{X}_0 \otimes_{\Bbbk} \mathcal{Y}_0 & i = 0 \end{cases}$$

$$\partial_i^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{i+1}^{\mathcal{X}_{\geq 1} \otimes \mathcal{Y}_{\geq 1}} & i \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & i = 1 \end{cases}$$

Thm (G):

If $\mathcal{S}$ and $\mathcal{Z}$ are minimal resolutions of $\Bbbk[\underline{x}]/I$ and $\Bbbk[\underline{y}]/J$, then $\mathcal{S} * \mathcal{Z}$ is a minimal resolution of $\dfrac{\Bbbk[\underline{x}, \underline{y}]}{\langle I \cdot J \rangle}$.

Thm (G)

If $\mathcal{S}$ is a DGA resolution and $\mathcal{T}$ is a Taylor resolution, then $\mathcal{S} * \mathcal{T}$ is a DGA with product

$$
\begin{aligned}
(e_M * f_\Omega)(e_N * f_\Gamma)
&= \mathbb{1}[\omega_1 \le \gamma_1 < \omega_2]\,(-1)^{(|e_N|-1)(|f_\Omega|-1)}\, e_M e_N * P_1(f_\Omega) f_\Gamma \\
&\quad - \mathbb{1}[\omega_1 \le \gamma_1]\,\mathbb{1}[|e_N| = 1]\; e_M\, \partial^x(e_N) * f_\Omega f_\Gamma \\
&\quad + \mathbb{1}[\gamma_1 < \omega_1 < \gamma_2]\,(-1)^{|e_N|(|f_\Omega|-1)}\, e_M e_N * f_\Omega P_1(f_\Gamma) \\
&\quad - \mathbb{1}[\gamma_1 < \omega_1]\,\mathbb{1}[|e_M| = 1]\,(-1)^{(|e_N|-1)|f_\Omega|}\, \partial^x(e_M)\, e_N * f_\Omega f_\Gamma
\end{aligned}
$$

where $\Omega = \{\omega_1 < \omega_2 < \cdots\}$, $\Gamma = \{\gamma_1 < \gamma_2 < \cdots\}$, and $P_1(f_\Omega) = \dfrac{m_\Omega}{m_{\Omega \setminus \omega_1}} f_{\Omega \setminus \omega_1}$.

Corollary:

The fiber product $k[\underline{x}] \times_k k[\underline{y}]$ is minimally resolved by the DGA $\mathcal{X} * \mathcal{Y}$.

Case 2: $\dfrac{k[\underline{x}]}{I} \times_k k[\underline{y}]$, $\quad I \subseteq \langle \underline{x} \rangle^2$

Idea: Build a mapping cone such that the corresponding L.E.S. in homology reduces to

$$0 \longrightarrow I \cdot \frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y} \rangle} \longrightarrow \frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\cdot\underline{y} \rangle} \longrightarrow \frac{k[\underline{x},\underline{y}]}{\langle I, \underline{x}\cdot\underline{y} \rangle} \longrightarrow 0$$

$$\Phi : \Sigma^{-1}(\mathcal{S}_{\geq 1} \otimes \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$$

<u>Need</u>: $\phi : \mathcal{S} \longrightarrow \mathcal{X}$

$$(x_1^2 \ \ x_1x_2) \subseteq \langle \underline{x} \rangle^2$$

<u>Example</u>:

$$\begin{tikzcd}
\mathcal{S} = 0 \arrow[r] \arrow[d, "\phi"'] & k[\underline{x}] \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"] \arrow[d, "x_1"] & k[\underline{x}]^2 \arrow[r, "(x_1^2 \ \ x_1x_2)"] \arrow[d, "\begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix}"] & k[\underline{x}] \arrow[r] \arrow[d, "1"] & 0 \\
\mathcal{X} = 0 \arrow[r] & k[\underline{x}] \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"'] & k[\underline{x}]^2 \arrow[r, "(x_1 \ \ x_2)"'] & k[\underline{x}] \arrow[r] & 0
\end{tikzcd}$$

Lift: $\Phi : \Sigma^{-1}(\mathcal{S}_{2} \otimes_{k} \mathcal{Y}) \longrightarrow \mathcal{X} * \mathcal{Y}$

Given by,

$$\Phi(\alpha \otimes f_\Gamma) = \mathbb{1}[|f_\Gamma| > 0](-1)^{|\alpha| + |f_\Gamma|} \phi(\alpha) * f_\Gamma + \mathbb{1}[|f_\Gamma| = 0]\,\mathbb{1}[|\alpha| = 1]\, \partial^{\mathcal{S}}_{1}(\alpha) * f_\Gamma$$

Thm (G)

The map Φ is a chain map and $\mathrm{Cone}(\Phi)$ minimally resolves $\frac{k[\underline{x}]}{I} \times_k k[\underline{y}]$ provided $\mathcal{S}$ is minimal and $I \subseteq \langle \underline{x} \rangle^2$.

Thm (G)

The action of $\mathcal{X} * \mathcal{Y}$ on $\mathrm{Cone}(\Phi)$ given by

$$(e_M * f_\Omega)(\alpha \otimes f_\Gamma) = -\mathbb{1}[|e_M| = 1](-1)^{(|\alpha|-1)|f_\Omega|} \partial^{\mathcal{X}}(e_M)\alpha \otimes f_\Omega f_\Gamma + \mathbb{1}[w_1 \leq \gamma_1](-1)^{|\alpha|(|f_\Omega|-1)+|f_\Gamma|} e_M \phi(\alpha) * f_\Omega f_\Gamma$$

makes $\mathrm{Cone}(\Phi)$ into a DG $\mathcal{X} * \mathcal{Y}$-module.

$\left(\mathcal{S} \text{ minimal}, I \subseteq \langle \underline{x} \rangle^2, \phi_0 = 1\right)$

Conj: If $\mathcal{S}$ is DGA, ϕ a DG-morphism then $\mathrm{Cone}(\Phi)$ is a DGA $(\alpha \otimes f_\Omega)(\beta \otimes f_\Gamma) = (-1)^{|\beta||f_\Omega|} \alpha\beta \otimes f_\Omega f_\Gamma$