Semidualizing Modules Over Numerical Semigroup Rings

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Motivating Goal:

Goal: Classify which numerical semigroup rings possess a nontrivial semidualizing module.

Questions to address:

- What is a numerical semigroup ring?
- What is a semidualizing module? What makes it nontrivial?
- What is the motivation behind this goal?
- What progress has been made?

Numerical Semigroup Rings

Definition (Numerical Semigroup)

Let $\mathbb{N}$ *denote the set of non-negative integers. A* **numerical semigroup** *is a subset* $H \subset \mathbb{N}$ *such that*

1. $0 \in H$;
2. H *is closed under addition; and*
3. $\gcd(H) = 1$.

Definition (Numerical Semigroup Ring)

Let k *be a field and* $H = \langle a_1, \ldots, a_\ell \rangle$ *a numerical semigroup. The* **numerical semigroup ring** *(associated to* H *over* k*) is the ring*

$$R_H = k[\![H]\!] := k[\![t^{a_1}, \ldots, t^{a_\ell}]\!] \subseteq k[\![t]\!] \,.$$

Numerical Semigroup Terminology and Facts

Set $H = \langle a_1, \ldots, a_\ell \rangle$ with $a_1 < a_2 < \cdots < a_\ell$.

1. **Frobenius of H:**

$$F_H := \max \mathbb{N} \backslash H$$

2. **Pseudo-Frobenius number of H:**

$$\mathrm{PF}(H) := \{f \in \mathbb{N} \backslash H : \text{for all } a \in H, a + f \in H\}$$

3. **Multiplicity of R_H:**

$$e(R_H) = a_1$$

4. **Embedding Dimension of R_H:**

$$\mathrm{edim}(R_H) = \ell$$

Example: $H = \langle 9, 12, 15, 17, 19 \rangle$

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53

1. $F_H = 25$
2. $\mathrm{PF}(H) = \{20, 22, 23, 25\}$
3. $e(R_H) = 9$
4. $\mathrm{edim}(R_H) = 5$

Canonical Module

Definition

Let $(R, \mathfrak{M})$ be a Cohen-Macaulay local ring and K an R-module. We say K is a **canonical module** *if it is*

1. *a maximal Cohen-Macaulay of type 1; and*
2. *K has finite injective dimension.*

Fact

Let $(R, \mathfrak{M})$ be a Cohen-Macaulay local ring. The canonical module K is unique up to isomorphism.

Fact

Let H be a numerical semigroup. The ring R_H possesses a canonical module.

Example Continued: $H = \langle 9, 12, 15, 17, 19 \rangle$

Fact

Let H be a numerical semigroup and R_H the corresponding numerical semigroup ring with canonical module K_H. We have

$$K_H \cong \left\langle t^{F_H - f} : f \in \mathrm{PF}(H) \right\rangle .$$

Recall: $\mathrm{PF}(H) = \{20, 22, 23, 25 = F_H\}$

$$K_H \cong \langle t^{25-25}, t^{25-23}, t^{25-22}, t^{25-20} \rangle$$
$$= \langle 1, t^2, t^3, t^5 \rangle$$

Semidualizing Module

Let K be the canonical module for $(R, \mathfrak{M})$.

Definition

A finitely generated R-module C is **semidualizing** *if it satisfies*

1. *The natural homothety map $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$ is an R-module isomorphism; and*
2. $\mathrm{Ext}_R^i(C, C) = 0$ *for all $i > 0$.*

It follows that R and K are both semidualizing module for R; we refer to them as **trivial semidualizing modules***.*

Fact (Christensen'01)

If R is a Gorenstein, then it only has trivial semidualizing modules.

Only Trivial Semidualizing Modules

Proposition

Let $(R, \mathfrak{M})$ *be a Cohen-Macaulay local ring such that*

1. $\operatorname{edim}(R) - \operatorname{depth}(R) \leq 3$ *(LM'20, AINSW'22);*
2. $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{M}$ *is* R*-regular and* $R/(\mathbf{x})$ *only has trivial semidualizing module (CSW'08 and FSW'07, NSW'13);*
3. $e(R) \leq 8$*; or*
4. $R \cong S/I$ *where* S *is regular local and* I *is a Burch ideal of* S*;*

then R *only has trivial semidualizing modules.*

Proposition

Let $(R, \mathfrak{M})$ *be a Cohen-Macaulay local ring. If* $e(R) = 9$ *and* R *has a nontrivial semidualizing module, then* R *has type* $r(R) = 4$ *and* $\operatorname{edim}(R) = 4 + \dim R$*.*

Example Continued: $H = \langle 9, 12, 15, 17, 19 \rangle$

Observe:

- $e(R_H) = 9$
- $K_H \cong \langle 1, t^2, t^3, t^5 \rangle$
- $r(R_H) = \mu_R(K_H) = 4$
- $\operatorname{edim}(R_H) = 5 = 4 + \dim R_H$

Fact: $I = \langle 1, t^2 \rangle$ and $I^\vee = \langle 1, t^3 \rangle$ are semidualizing over R_H.

Lemma

Let R be a Cohen-Macaulay local ring with canonical module K. Write the functor $\operatorname{Hom}_R(-, K)$ as $(-)^\vee$. Let C be a semidualizing over R. Then

1. *$C^\vee$ is semidualizing.*
2. *$C \otimes_R C^\vee \cong K$.*
3. *$\mu_R(C)\mu_R(C^\vee) = r(R)$.*

Set-up:

Let H be a numerical semigroup such that:

1. $e(R_H) = 9$; and
2. $\mathrm{edim}(R_H) = 5$.

We consider $H = \langle 9, a, b, c, d \rangle$ with $9 < a < b < c < d$.

Problem: There are infinitely many choices of H!

Solution: Equivalence classes

Apéry Set

Definition

Given a numerical semigroup H and integer $3 \leq m \in H$, the **Apéry set of H with respect to m** *is the set*

$$\mathrm{Ap}_m(H) = \{0, h_1, h_2, \ldots, h_{m-1}\}$$

where $h_i = \min\{h \in H : h \equiv i \mod m\}$ *for* $1 \leq i < m$.

Consider $H = \langle 9, 12, 15, 17, 19 \rangle$

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35

$$\mathrm{Ap}_9(H) = \{0, 19, 29, 12, 31, 32, 15, 34, 17\}$$

Apéry Set Relations for $H = \langle 9, 12, 15, 17, 19 \rangle$

$$\mathrm{Ap}_9(H) = \{0, 19, 29, 12, 31, 32, 15, 34, 17\}$$

Relations:

$$h_1 + h_3 = h_4, \quad h_1 + h_6 = h_7, \quad h_3 + h_8 = h_2,$$

$$h_6 + h_8 = h_5, \quad h_8 + h_8 = h_7, \quad \begin{cases} h_i + h_j > h_{i+j} & i + j < 9 \\ h_i + h_j > h_{i+j-9} & i + j > 9 \end{cases}$$

Fact

Let H be a numerical semigroup and $m \in H$. Given $1 \leq i, j < m$ with $i + j \neq m$, then the elements of $\mathrm{Ap}_m(H)$ satisfy

$$\begin{cases} h_i + h_j \geq h_{i+j} & i + j < m \\ h_i + h_j \geq h_{i+j-m} & i + j > m \end{cases}.$$

Kunz's Polyhedron (Kind of)

For $3 \leq m \in \mathbb{Z}$, the polyhedral cone C_m is the solution set of

$$\begin{cases} X_i + X_j \geq X_{i+j} & 1 \leq i < j < m \text{ and } i+j < m \\ X_i + X_j \geq X_{i+j-m} & 1 \leq i < j < m \text{ and } i+j > m \end{cases}.$$

Note: The facets of C_m are given by

$$E_{ij} = \begin{cases} X_i + X_j = X_{i+j} & 1 \leq i < j < m \text{ and } i+j < m \\ X_i + X_j = X_{i+j-m} & 1 \leq i < j < m \text{ and } i+j > m \end{cases}.$$

If F is a face of C_m, then it is completely determined by the set

$$\Delta_F := \{(i,j) : F \subseteq E_{ij}\}.$$

C_m and $\mathrm{Ap}_m(H)$

Fact

If H is a numerical semigroup with $m \in H$, then $\mathrm{Ap}_m(H)$ is a solution set for the defining equations of C_m. Consequently, we can associated H with a face of C_m via $\mathrm{Ap}_m(H)$.

Example: $H = \langle 9, 12, 15, 17, 19 \rangle$ has relations

$$h_1 + h_3 = h_4, \quad h_1 + h_6 = h_7, \qquad h_3 + h_8 = h_2,$$
$$h_6 + h_8 = h_5, \quad h_8 + h_8 = h_7, \quad \begin{cases} h_i + h_j > h_{i+j} & i + j < 9 \\ h_i + h_j > h_{i+j-9} & i + j > 9 \end{cases}$$

We associate H with the face defined by

$$\Delta = \{(1,3), (1,6), (3,8), (6,8), (8,8)\}.$$

Equivalence Classes

Let $\sigma \in \operatorname{Aut}(\mathbb{Z}/m\mathbb{Z})$ and given a face Δ, define

$$\sigma(\Delta) := \{(\sigma(i), \sigma(j)) : (i,j) \in \Delta\}.$$

Proposition (C-K'23)

Let $\sigma \in \operatorname{Aut}(\mathbb{Z}/m\mathbb{Z})$. Suppose H and H' are numerical semigroups associated to Δ and $\sigma(\Delta)$, respectively. The ring R_H has a nontrivial semidualizing module if and only if $R_{H'}$ has a nontrivial semidualizing module.

Class Representatives?

Consider the case $m = 9$ and $H = \langle 9, a, b, c, d \rangle$.

1	2	4	5	7	8
$(1,2,3,4)$	$(2,4,6,8)$	$(3,4,7,8)$	$(1,2,5,6)$	$(1,3,5,7)$	$(5,6,7,8)$
$(1,2,3,5)$	$(1,2,4,6)$	$(2,3,4,8)$	$(1,5,6,7)$	$(3,5,7,8)$	$(4,6,7,8)$
$(1,2,3,6)$	$(2,3,4,6)$	$(3,4,6,8)$	$(1,3,5,6)$	$(3,5,6,7)$	$(3,6,7,8)$
$(1,2,3,7)$	$(2,4,5,6)$	$(1,3,4,8)$	$(1,5,6,8)$	$(3,4,5,7)$	$(2,6,7,8)$
$(1,2,3,8)$	$(2,4,6,7)$	$(3,4,5,8)$	$(1,4,5,6)$	$(2,3,5,7)$	$(1,6,7,8)$
$(1,2,4,5)$	$(1,2,4,8)$	$(2,4,7,8)$	$(1,2,5,7)$	$(1,5,7,8)$	$(4,5,7,8)$
$(1,2,4,7)$	$(2,4,5,8)$	$(1,4,7,8)$	$(1,2,5,8)$	$(1,4,5,7)$	$(2,5,7,8)$
$(1,2,6,7)$	$(2,3,4,5)$	$(1,4,6,8)$	$(1,3,5,8)$	$(2,5,6,7)$	$(2,3,7,8)$
$(1,2,6,8)$	$(2,3,4,7)$	$(4,5,6,8)$	$(1,3,4,5)$	$(2,5,6,7)$	$(1,3,7,8)$
$(1,2,7,8)$	$(2,4,5,7)$	$(1,4,5,8)$	$(1,4,5,8)$	$(2,4,5,7)$	$(1,2,7,8)$
$(1,3,4,6)$	$(2,3,6,8)$	$(3,4,6,7)$	$(2,3,5,6)$	$(1,3,6,7)$	$(3,5,6,8$
$(1,3,4,7)$	$(2,5,6,8)$	$(1,3,4,7)$	$(2,5,6,8)$	$(1,3,4,7)$	$(2,5,6,8)$
$(1,3,6,8)$	$(2,3,6,7)$	$(3,4,5,6)$	$(3,4,5,6)$	$(2,3,6,7)$	$(1,3,6,8)$
$(1,4,6,7)$	$(2,3,5,8)$	$(1,4,6,7)$	$(2,3,5,8)$	$(1,4,6,7)$	$(2,3,5,8)$

Class Representatives and Results

Fact: For $m = 9$ and $H = \langle 9, a, b, c, d\rangle$ there are 127 classes.

$(\bar{a}, \bar{b}, \bar{c}, \bar{d})$	Δ	Sample	Burch?	Nontrivial?
$(1,3,6,8)$	$\{(1,1),(1,3),(6,8),(1,6)\}$	$\langle 9,10,15,17,21\rangle$	Yes	No
$(1,3,6,8)$	$\{(1,1),(1,3),(6,8),(8,8)\}$	$\langle 9,17,19,24,30\rangle$	Yes	No
$(1,3,6,8)$	$\{(3,8),(1,3),(6,8),(1,6)\}$	$\langle 9,12,15,26,28\rangle$	No	Yes
$(1,3,6,8)$	$\{(3,8),(1,3),(6,8),(8,8)\}$	$\langle 9,12,15,17,28\rangle$	Yes	No
$(1,3,6,8)$	$\{(1,1),(1,3),(6,8),(1,6),(8,8)\}$	$\langle 9,24,26,28,39\rangle$	Yes	No
$(1,3,6,8)$	$\{(1,1),(3,8),(1,3),(6,8),(1,6)\}$	$\langle 9,12,15,19,26\rangle$	No	Yes
$(1,3,6,8)$	$\{(1,1),(3,8),(1,3),(6,8),(8,8)\}$	$\langle 9,17,19,21,24\rangle$	Yes	No
$(1,3,6,8)$	$\{(3,8),(1,3),(6,8),(1,6),(8,8)\}$	$\langle 9,12,15,17,19\rangle$	No	Yes
$(1,3,6,8)$	$\{(1,1),(3,8),(1,3),(6,8),(1,6),(8,8)\}$	$\langle 9,15,17,19,21\rangle$	No	Yes
$(1,4,6,7)$	$\{(1,1),(6,6),(7,7),(4,4)\}$	$\emptyset$	-	-
$(1,2,3,6)$	$\{(1,3),(2,3),(1,6),(2,6)\}$	$\langle 9,12,15,19,20\rangle$	No	Yes
$(1,2,3,6)$	$\{(1,3),(2,2),(2,3),(1,6),(2,6)\}$	$\langle 9,10,11,12,15\rangle$	No	Yes
$(1,2,4,7)$	$\{(1,2),(1,4),(7,7),(2,4),(1,7)\}$	$\langle 9,19,20,25,31\rangle$	No	No

Remark: There are 24 distinct Δ.

Main Result 1

Theorem (C-K'23)

Let $H = \langle 9, a, b, c, d \rangle$ be a numerical semigroup associated with Δ. The ring R_H has a nontrivial semidualizing module if and only if there exists $\sigma \in \operatorname{Aut}(\mathbb{Z}/9\mathbb{Z})$ such that $\sigma(\Delta)$ is equal to one of the following sets;

1. $\{(1,3),(2,3),(1,6),(2,6)\}$; size of orbit $= 6$
2. $\{(1,3),(2,2),(2,3),(1,6),(2,6)\}$;
3. $\{(1,1),(3,8),(1,3),(6,8),(1,6)\}$;
4. $\{(3,8),(1,3),(6,8),(1,6)\}$; size of orbit $= 3$
5. $\{(1,1),(3,8),(1,3),(6,8),(1,6),(8,8)\}$.

Higher Multiplicity

Question: How do these results carry to numerical semigroups H where $e(R_H) > 9$?

Issue: If R_H has a nontrivial semidualizing module and $e(R_H) = 10$, then $\operatorname{edim}(R_H) \in \{5, 6\}$.

Example $e(R_{H'}) > 9$

Audience Participation: Pick a highlighted number.

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35

Example: $H' = \langle 26, 27, 36, 45, 51, 57 \rangle$

$e(R_{H'}) = 26$

Semidualizing Module: $\langle 1, t^6 \rangle$

Gluings

Definition

Let H_1 and H_2 be numerical semigroups. Given $a_i \in H_i$ such that $\gcd(a_1, a_2) = 1$, the gluing of H_1 and H_2 (with respect to a_1 and a_2) is the numerical semigroup

$$H = \langle a_2 H_1, a_1 H_2 \rangle = \{a_2 r + a_1 s : r \in H_1, s \in H_2\}.$$

Moreover, if we write R_i for R_{H_i}, then

$$R_H = k\left[\left[t^{a_2 r + a_1 s} : t^r \in R_1, t^s \in R_2\right]\right].$$

Example continued:

$H' = \langle 26(1), 3H \rangle$ where $H = \langle 9, 12, 15, 17, 19 \rangle$

Gluing Results

Theorem (C-K'23)

Let H_1 and H_2 are numerical semigroups, take $a_i \in H_i$ such that $\gcd(a_1, a_2) = 1$, and set $H = \langle a_2 H_1, a_1 H_2 \rangle$. Suppose R_i has semidualizing module I_i, then R_H has semidualizing module

$$I = \langle t^{a_2 r + a_1 s} : t^r \in I_1, t^s \in I_2 \rangle.$$

Corollary (C-K'23)

If either I_1 or I_2 is nontrivial, then I is a nontrivial semidualizing module over R_H.

Example continued: $H = \langle 9, 12, 15, 17, 19 \rangle$ has $I = \langle 1, t^2 \rangle$

so $H' = \langle 26, 27, 36, 45, 51, 57 \rangle$ has $I' = \langle 1^3, (t^2)^3 \rangle = \langle 1, t^6 \rangle$

Main Result 2

Theorem (C-K'23)

For all $a \in \mathbb{Z}$ with $a \geq 9$, there exists a local ring R with $e(R) = a$ such that R has a nontrivial semidualizing module.

Proof.

For each $a \geq 9$, we give a numerical semigroup H. The gluing $H' = \langle a, bH \rangle$ with $9b \geq a$ where $\gcd(a, b) = 1$ produces $R = R_{H'}$.
Case 1: For $a \notin \{13, 14, 16, 17\}$, consider $H = \langle 9, 10, 11, 12, 15 \rangle$.
Case 2: For $a = 13$, consider $H = \langle 9, 11, 12, 13, 15 \rangle$.
Case 3: For $a \in \{14, 16\}$, consider $H = \langle 9, 12, 14, 15, 16 \rangle$.
Case 4: For $a = 17$, consider $H = \langle 9, 12, 15, 17, 19 \rangle$. □

Thank you!