DG-Structures on Minimal Free Resolutions of Fiber Products

S, T rings with ring homs

$$\pi_S : S \longrightarrow k$$
$$\pi_T : T \longrightarrow k$$

$$S \times_k T = \{ (s,t) \in S \times T \mid \pi_S(s) = \pi_T(t) \}.$$

Thm (N sw):

Let M and N be finitely generated modules over $R = S \times_k T$. If

$$\mathrm{Tor}_i^R(M,N) = 0 = \mathrm{Tor}_{i+1}^R(M,N) \quad i \geq 5$$

then $\mathrm{pd}_R(M) \leq 1$ or $\mathrm{pd}_R(N) \leq 1$.

Thm (AINSW)

Let R be a local ring. Assume there exists a minimal Cohen presentation $\hat{R} \cong P/I$ satisfying

a.) Some minimal free resolution of $\hat{R}$ over P has a structure of a DG Algebra;

b.) other conditions.

If M and N are finite R-modules and $\operatorname{Tor}^R(M,N)$ is bounded, then M or N has finite projective dimension.

$A = k[\![\underline{x}]\!] \supseteq I$, $B = k[\![\underline{y}]\!] \supseteq J$, $C = k[\![\underline{x}, \underline{y}]\!]$

$$\frac{A}{I} \times_k \frac{B}{J} \cong \frac{C}{\langle I, \underline{x}\underline{y}, J \rangle}$$

First: $I = 0 = J$, $\quad k[\![\underline{x}]\!] \times_k k[\![\underline{y}]\!] \cong \dfrac{k[\![\underline{x},\underline{y}]\!]}{\langle \underline{x}\,\underline{y}\rangle}$

Relates to edge ideal of $K_{m,n}$

$\underline{x} = x_1, \dots, x_m$

$\underline{y} = y_1, \dots, y_n$

Visscher '06 gives an explicit minimal resolution $\dfrac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle}$ over $k[\underline{x},\underline{y}]$.

A, B, C local rings

$\tilde{I} \subseteq A$ ideal

$\tilde{J} \subseteq B$ ideal

Flat ring homomorphisms $A \to C \leftarrow B$

Take min res'L X of $A/\tilde{I}$, Y of $B/\tilde{J}$

Construct a resolution $X * Y$ of $\dfrac{C}{\tilde{I}\tilde{J}}$

$\hat{X} := X \otimes_A C$ resolve $C/\tilde{I}$

$\hat{\mathcal{Y}} := \mathcal{Y} \otimes_B C$ resolve $C/\tilde{J}$

$$(\mathcal{X} * \mathcal{Y})_i = \begin{cases} \left(\hat{\mathcal{X}}_{\geq 1} \otimes_C \hat{\mathcal{Y}}_{\geq 1}\right)_{i+1} & i \geq 1 \\ \mathcal{X}_0 \otimes_C \mathcal{Y}_0 & i = 0 \end{cases}$$

$$\partial_i^{\mathcal{X} * \mathcal{Y}} = \begin{cases} \partial_{i+1}^{\hat{\mathcal{X}}_{\geq 1} \otimes \hat{\mathcal{Y}}_{\geq 1}} & i \geq 2 \\ \partial_1^{\hat{\mathcal{X}}} \otimes_C \partial_1^{\hat{\mathcal{Y}}} & i = 1 \end{cases}$$

Thm(-): Suppose $\operatorname{Tor}_i^C\left(\frac{C}{\tilde{I}}, \frac{C}{\tilde{J}}\right) = 0 \quad i \geq 1$, then $\mathcal{X} * \mathcal{Y}$ resolves $\frac{C}{\tilde{I}\tilde{J}}$ over C. Moreover if $\mathcal{X}, \mathcal{Y}$ are minimal, then $\mathcal{X} * \mathcal{Y}$ is minimal.

Cor(-):

Let $\mathcal{X}$ be the Koszul complex on $\underline{x}$ over $k[[\underline{x}]]$, let $\mathcal{Y}$ be Koszul on $\underline{y}$ $k[[\underline{y}]]$. Then $\mathcal{X} * \mathcal{Y}$ minimally resolves

$$k[[\underline{x}]] \times_k k[[\underline{y}]].$$

Example: $k[[x_1, x_2]] \times_k k[[y_1, y_2, y_3]]$

$$\mathcal{X} = K^{k[[x]]}(x_1, x_2)$$

$$Y = K^{k[[y]]}(y_1, y_2, y_3)$$

$$
\begin{tikzcd}[column sep=huge]
0 \arrow[r] & R \arrow[r, "\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}", "\begin{pmatrix} \partial_3^{\hat{Y}} \\ \partial_2^{\hat{X}} \end{pmatrix}"'] & R^5 \arrow[r, "\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & +y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & +y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}", "\begin{pmatrix} \partial_2^{\hat{Y}} & 0 & 0 \\ -x_2 I_3 & \partial_3^{\hat{Y}} & 0 \\ x_1 I_3 & 0 & -\partial_3^{\hat{Y}} \end{pmatrix}"'] & R^9 \arrow[r, "\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix}"] & R^6 \arrow[r] & R \arrow[r] & 0
\end{tikzcd}
$$

$$\partial_2^{\mathcal{X}} \otimes I_3$$

$x_1 y_1, \ x_1 y_2 \ \ldots$

What is a DGA?

An R-complex $\mathcal{X}$ equiped with a binary operation $\mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ such that ~~if~~ is

- unital
- graded commutative: $ab = (-1)^{|a||b|} ba$

  $a^2 = 0$ if $|a|$ is odd
- associative
- satisifies the Leibniz rule

$$\partial^{\mathcal{X}}_{|a|+|b|}(ab) = \partial^{\mathcal{X}}_{|a|}(a)b + (-1)^{|a|} a\, \partial^{\mathcal{X}}_{|b|}(b).$$

Examples:

1.) $\mathcal{X} = K^{k[[x_1, x_2]]}(x_1, x_2)$ Koszul Complex

e_ϕ, e_1, e_2, e_{12} basis elements

e_ϕ is the identity

$e_1 e_2 = -e_2 e_1 = e_{12}$

2.) $Y = K^{R[[y_1, y_2, y_3]]}(y_1, y_2, y_3)$ Koszul

$\Omega, \Gamma \subseteq \{1,2,3\}$

$$f_\Omega f_\Gamma = \begin{cases} \text{sign}(\Omega,\Gamma)\, f_{\Omega\cup\Gamma} & \Omega\cap\Gamma = \phi \\ 0 & \Omega\cap\Gamma \neq \phi. \end{cases}$$

3.) $X * Y$ the DG structure is elusive

$$(e_1 * f_1)(e_1 * f_2) = e_1^2 * y_1 f_2 - x_1 e_1 * f_1 f_2$$
$$= -x_1 e_1 * f_{12}$$

$$(e_1 * f_1)(e_1 * f_{12}) = e_1^2 * y_1 f_{12} - x_1 e_1 * f_1 f_{12}$$
$$= 0$$

$$(e_1 * f_1)(e_2 * f_2) = e_1 e_2 * y_1 f_2 - x_2 e_1 * f_1 f_2$$
$$= y_1 e_{12} * f_2 - x_2 e_1 * f_{12}$$

Thm(–): Suppose $\mathcal{X}$ is any DG A-algebra ← the ring A

and $\mathcal{Y}$ a Koszul complex over B. Then

$\mathcal{X} * \mathcal{Y}$ is a DG algebra over C with

products defined by the following

$$
\begin{aligned}
(e_M * f_\Omega)(e_N * f_\Gamma) = {} & \mathbb{1}[w_{a_1} \le \gamma_{b_1} < w_{a_2}](-1)^{(|f_\Omega|-1)(|e_N|-1)} e_M e_N * P_1(f_\Omega) f_\Gamma \\
& - \mathbb{1}[w_{a_1} \le \gamma_{b_1}]\mathbb{1}[|e_N| = 1] \; e_M \partial^{\mathcal{X}}(e_N) * f_\Omega f_\Gamma \\
& + \mathbb{1}[\gamma_{b_1} < w_{a_1} < \gamma_{b_2}](-1)^{(|f_\Omega|-1)|e_N|} e_M e_N * f_\Omega P_1(f_\Gamma) \\
& - \mathbb{1}[\gamma_{b_1} < w_{a_1}]\mathbb{1}[|e_M| = 1](-1)^{|f_\Omega|(|e_N|-1)} \partial^{\mathcal{X}}(e_M) e_N * f_\Omega f_\Gamma .
\end{aligned}
$$

Cor(–):

If $\mathcal{X}$ is Koszul on $\underline{x}$, $\mathcal{Y}$ on $\underline{y}$,

then $\mathcal{X} * \mathcal{Y}$ is a minimal, DG algebra

resolution of $k[[\underline{x}]] \times_k k[[\underline{y}]]$ over $k[[\underline{x}, \underline{y}]]$.

What if $I \neq 0$?

Take a minimal resolution of $\frac{k[[\underline{x}]]}{I}$

$I \subseteq \langle \underline{x} \rangle^2$

$$\begin{tikzcd} \frac{k[[\underline{x}]]}{I} \arrow[r] & k \cong \frac{k[[\underline{x}]]}{\langle \underline{x} \rangle} \\ \text{resolved by } \mathcal{S} \arrow[u] & X \arrow[u] \end{tikzcd}$$

Lift to a chain map

$$\phi : \mathcal{S} \longrightarrow X$$

$$0 \to I \frac{k[[\underline{x}, \underline{y}]]}{\langle \underline{x}\underline{y} \rangle} \longrightarrow \frac{k[[\underline{x}, \underline{y}]]}{\langle \underline{x}\underline{y} \rangle} \longrightarrow \frac{k[[\underline{x}, \underline{y}]]}{\langle I, \underline{x}\underline{y} \rangle} \to 0$$

<u>Note</u>: $I \cdot \frac{k[[\underline{x}, \underline{y}]]}{\langle \underline{x}\underline{y} \rangle} \cong I \otimes_{k[[\underline{x}, \underline{y}]]} \frac{k[[\underline{x}, \underline{y}]]}{\langle \underline{y} \rangle}$

$$\hat{\mathcal{S}} = \mathcal{S} \otimes_{k[[x]]} k[[x, y_-]] \qquad \uparrow \qquad\qquad \uparrow$$

$$\hat{\mathcal{S}}_{\geq 1} \qquad \text{resolved by } \hat{\gamma}$$

$$\Phi : \Sigma^{-1}\left(\hat{\mathcal{S}}_{\geq 1} \otimes \tilde{\gamma}\right) \longrightarrow \mathcal{X} * \tilde{\gamma}$$

$$\Phi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha| + |\beta|} \phi(\alpha) * \beta & |\beta| > 0 \\ \partial_1^{\vec{\mathcal{S}}}(\alpha) * \beta & |\beta| = 0, \ |\alpha| = 1 \\ 0 & |\beta| = 0 \ \ |\alpha| > 1 \end{cases}$$

Prop (-): Φ is a chain map

Thm (-): Cone (Φ) is a resolution of

$$\frac{k[[x, y_-]]}{\langle I, x y_- \rangle} = \frac{k[[x]]}{I} \times_k k[[y_-]].$$

<u>Thm (-):</u> Cone(Φ) is a DG $X * Y$-module under the action:

$$(e_I * f_\Omega)(\alpha \otimes f_\Gamma) = -\mathbb{1}[|e_I| = 1](-1)^{(|\alpha|-1)|f_\Omega|} \partial^X(e_I)\alpha \otimes f_\Omega f_\Gamma + \mathbb{1}[w_{a_1} \leq \partial_{b_1}](-1)^{\square} e_I \phi(\alpha) * f_\Omega f_\Gamma.$$