DG-Algebra Resolutions for products of ideals

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- Notation
- Fiber Products
- Nasseh, Sather-Wagstaff '17

  "Fiber Products are Tor-friendly"
  - Uses minimal resolutions over the fiber product
  - Can we prove this with other methods?
    - other rings have been shown to be Tor-friendly using DG-algebra methods
- Avramov, Iyengar, Nasseh, Sather-Wagstaff '19

  If some minimal resolution of R/I over R has a DG structure, plus some other conditions, then R/I is Tor-friendly.
  - This why I like to think of $S \times_k T$ as

    $R/\langle I, J, xy \rangle$.
- DG-Algebra definition / DG-algebra resolution
  - Let A be an R complex with a binary operation $A \times A \longrightarrow A$ with following properties for all $a, b, c \in A$
    - $(ab)c = a(bc)$
    - $(a+b)c = ac + bc$ if $|a| = |b|$
    - $\exists\, 1_A \in A_0$ such that $1_A x = x \quad \forall x \in A$

- $ab = (-1)^{|a||b|} ba \in A_{|a|+|b|}$ and $a^2 = 0$ if $|a|$ is odd
- $\partial_{|a|+|b|}(ab) = \partial_{|a|}(a)b + (-1)^{|a|} a \partial_{|b|}(b)$

- Examples:
  - Koszul complex with exterior algebra multiplication
    - Note: I will write $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell} \in R^{\binom{n}{\ell}}$ as $e_{\{i_1, i_2, \dots, i_\ell\}}$ with $i_1, i_2, \dots, i_\ell \in [n] = \{1, \dots, n\}$
  - Taylor resolution
    - Used for monomial ideals
    - looks like the Koszul complex, is Koszul for regular sequences
    - same format for multiplication

$$e_I \cdot e_J = \begin{cases} \underline{\ ?\ } \operatorname{sign}(I,J) e_{I\cup J} & I \cap J = \phi \\ 0 & I \cap J \neq \phi \end{cases}$$

    - not always minimal
  - If there's time, mention $T^R(x^a, y^b, xy)$
    - DGA-res'l for $k[x]/\langle x^a \rangle \times_k k[y]/\langle y^b \rangle$
    - non-minimal
    - minimal can be given a DG structure using Hilbert-Burch, which can be realized as Taylor mod a DG-ideal.

Fact: Every R-algebra has a DG-algebra resolution over R.

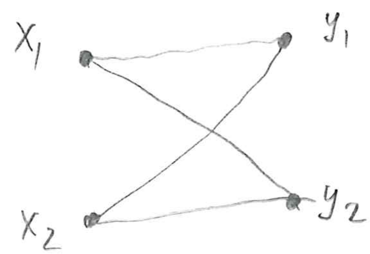
Question: When is the DGA-res'l minimal?

- Visscher '06
  - $I_{K_{e,d}} = \langle x_i y_j \mid 1 \le i \le e, 1 \le j \le d \rangle$
  - $R/I_{K_{e,d}}$ has an explicit minimal resolution supported on a cell complex.

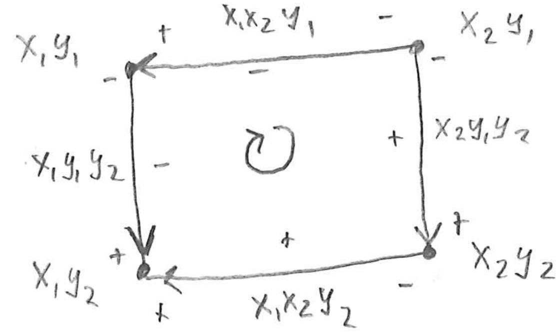
- Make use of Bayer and Sturmfels '98 <u>On Cellular Resolutions of Monomial Modules</u>
- Unpublished paper posted on the ArXiv in 1/16 claims to prove these resolutions have a DG structure using discrete Morse theory
  - agree with products ab with $a, b \in A_1$.
  - there's an issue with R-linearity in higher degree.

<u>Example</u>: $K_{2,2}$

<u>Graph</u>:



Cell-complex



$$K: \quad 0 \to R \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} -x_2 & 0 & y_2 & 0 \\ x_1 & 0 & 0 & y_2 \\ 0 & -x_2 & -y_1 & 0 \\ 0 & x_1 & 0 & -y_1 \end{pmatrix}} R^4 \xrightarrow{(x_1y_1 \;\; x_2y_1 \;\; x_1y_2 \;\; x_2y_2)} R \to 0$$

$e_{x_1x_2y_1y_2}$

$e_{x_1x_2y_1}$, $e_{x_1x_2y_2}$, $e_{x_1y_1y_2}$, $e_{x_2y_1y_2}$

$e_{x_1y_1}$, $e_{x_2y_1}$, $e_{x_1y_2}$, $e_{x_2y_2}$

$$e_{x_1y_1} \cdot e_{x_2y_1} = y_1 e_{x_1x_2y_1}$$

$$e_{x_1y_1} \cdot e_{x_1y_2} = -x_1 e_{x_1y_1y_2}$$

$$e_{x_1y_1} \cdot e_{x_2y_2} = y_1 e_{x_1x_2y_2} - x_2 e_{x_1y_1y_2}$$

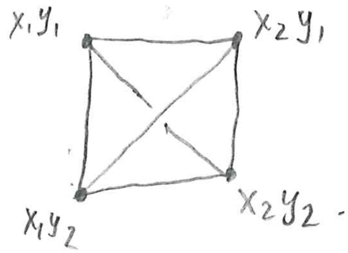
$$e_{x_1y_1} \cdot e_{x_1x_2y_1} = 0$$

$$e_{x_1y_1} \cdot e_{x_1y_1y_2} = 0$$

$$e_{x_1y_1} \cdot e_{x_1x_2y_2} = 0$$

$$e_{x_1y_1} \cdot e_{x_2y_1y_2} = y_1 e_{x_1x_2y_1y_2}$$

• Can be realized as $T^R(x_1y_1,\ x_2y_1,\ x_1y_2,\ x_2y_2)$ mod a DG-ideal, which kills the diagonals below (Can be seen explicitly in my Auburn talk)



- This requires making several, not necessarily intuitive choices
- Becomes messy for $K_{2,3}$ and $K_{3,3}$

• Can see Taylor/Koszul bits in the differentials; more apparent for $K_{2,3}$ and $K_{3,3}$.

- Construction:

• Let $\mathcal{X} = K^{R_x}(x_1, x_2)$, $\mathcal{Y} = K^{R_y}(y_1, y_2)$ and set

$$\mathcal{X}^+ = \quad 0 \longrightarrow \underset{2}{R_x} \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} \underset{1}{R_x^2} \longrightarrow \underset{0}{0} \longrightarrow 0$$

$\mathcal{Y}^+$ defined analogously.

• Define $\mathcal{X} *_R \mathcal{Y}$ via

$$(\mathcal{X} *_R \mathcal{Y})_\ell = \begin{cases} (\mathcal{X}^+ \otimes_R \mathcal{Y}^+)_{\ell+1} & \ell \geq 1 \\ R_x \otimes_R R_y & \ell = 0 \end{cases}$$

$$\partial_\ell^{\mathcal{X}*\mathcal{Y}} = \begin{cases} \partial_{\ell+1}^{\mathcal{X}^+ \otimes \mathcal{Y}^+} & \ell \geq 2 \\ \partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}} & \ell = 1 \end{cases}$$

• Observation: $K \cong \mathcal{X} *_R \mathcal{Y}$

Theorem:

If $\mathcal{X}$ is a free resolution for $R_x/\mathfrak{a}$ and $\mathcal{Y}$ is a free resolution for $R_y/\mathfrak{b}$, then $\mathcal{X} *_k \mathcal{Y}$ is a free resolution of

$$\frac{R_x \otimes_k R_y}{\mathfrak{a} \otimes_k \mathfrak{b}} \cong R/\langle \mathfrak{a}\mathfrak{b} \rangle.$$

Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are minimal, then so is $\mathcal{X} *_k \mathcal{Y}$.

Note: if $a \in \mathcal{X}_i$ and $b \in \mathcal{Y}_j$ then

$$|a * b| = i + j - 1 = |a \otimes b| - 1$$

Sketch of proof:

$\mathcal{X} * \mathcal{Y}$ a complex:

- $l \geq 2$, $\partial_l^{\mathcal{X}*\mathcal{Y}} \circ \partial_{l+1}^{\mathcal{X}*\mathcal{Y}} = \partial_{l+1}^{\mathcal{X}^+ \otimes \mathcal{Y}^+} \circ \partial_{l+2}^{\mathcal{X}^+ \otimes \mathcal{Y}^+} = 0$
- $l = 1$, $\partial_1^{\mathcal{X}*\mathcal{Y}} \circ \partial_2^{\mathcal{X}*\mathcal{Y}} = (\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}) \circ (\partial_3^{\mathcal{X}^+ \otimes \mathcal{Y}^+})$

$$= (\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}) \circ \left((\partial_2^{\mathcal{X}^+} \otimes 1) \oplus ((-1)^2 \otimes \partial_1^{\mathcal{Y}^+}) \oplus (\partial_1^{\mathcal{X}^+} \otimes 1) \oplus ((-1)^1 \otimes \partial_2^{\mathcal{Y}^+}) \right)$$

$$= (\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}) \left((\partial_2^{\mathcal{X}} \otimes 1) \oplus (-1 \otimes \partial_2^{\mathcal{Y}}) \right)$$

$$= 0$$

$\mathcal{X} * \mathcal{Y}$ acyclic:

- $l \geq 2$, $H_l(\mathcal{X} * \mathcal{Y}) = H_{l+1}(\mathcal{X} \otimes \mathcal{Y})$
- $H_1(\mathcal{X} * \mathcal{Y}) = \ker(\partial_1^{\mathcal{X}} \otimes \partial_1^{\mathcal{Y}}) / \operatorname{Im} \partial_2^{\mathcal{X}*\mathcal{Y}}$

- $H_0(X * Y) \cong \dfrac{R_x \otimes R_y}{\operatorname{Im} \partial_1^{X*Y}} = \dfrac{R_x \otimes R_y}{d \otimes g} \cong R/\langle dg \rangle$

- It turns out we can put a DG-structure on $X * Y$ as long Y is a Koszul complex or Taylor resolution
  - Need a function to do this
    - let $\Omega = \{w_1, \dots, w_\ell\} \subseteq \{1, \dots, \text{length}(Y)\}$ then f_Ω is a basis element of Y.
    - for $1 \le t \le \ell$ define
    $$P_t(f_\Omega) = \operatorname{proj}_{f_{\Omega \setminus \{w_t\}}}\left(\partial^Y_{|f_\Omega|}(f_\Omega)\right)$$
    - $\partial^Y(f_\Omega) = \displaystyle\sum_{t=1}^{|\Omega|} P_t(f_\Omega)$

Example:

$Y = K^R(y_1, \dots, y_6)$, $f_{\{1,3,6\}} \in Y_3$

- $P_1(f_{\{1,3,6\}}) = y_1 f_{\{3,6\}}$
- $P_2(f_{\{1,3,6\}}) = -y_3 f_{\{1,6\}}$
- $P_3(f_{\{1,3,6\}}) = y_6 f_{\{1,3\}}$

Theorem:

If X is a DG-algebra and Y is a Taylor resolution or Koszul complex, then $X *_R Y$ has a DG-structure given by

$$\begin{aligned}(e_i * f_\Omega)(e_j * f_\Gamma) = {} & \mathbb{1}[w_1 \le \gamma_1 < w_2]\,(-1)^{(|f_\Omega|-1)(|e_j|-1)}\,(e_i e_j * P_1(f_\Omega) f_\Gamma) \\ & - \mathbb{1}[w_1 \le \gamma_1]\ \mathbb{1}[|e_j| = 1]\,(e_i \partial^X(e_j) * f_\Omega f_\Gamma) \\ & + \mathbb{1}[\gamma_1 < w_1 < \gamma_2]\,(-1)^{(|f_\Omega|-1)|e_j|}\,(e_i e_j * f_\Omega P_1(f_\Gamma)) \\ & - \mathbb{1}[\gamma_1 < w_1]\,\mathbb{1}[|e_i| = 1]\,(\partial^X(e_i) e_j * f_\Omega f_\Gamma)\end{aligned}$$

where $\Omega = \{\omega_1, \omega_2, \ldots\} \subset \mathbb{N}$, $\Gamma = \{\theta_1, \theta_2, \ldots\} \subseteq \mathbb{N}$ (listed in ascending order), $\omega_2 = \infty$ if $\Omega = \{\omega_1\}$ (same for Γ), and

$$\mathbb{1}[A] = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

- Proof is roughly 30 pages
- indicator function is not essential but simplifies the proof
  - without indicator function, associativity takes 720 cases
- X has no restrictions
- Y needs to be supported on a (single) simplex and needs to have the same underlying structure as the exterior algebra;

$$f_\Omega f_\Gamma = \begin{cases} \underline{?}\ \mathrm{sign}(\Omega,\Gamma) f_{\Omega\cup\Gamma}, & \Omega\cap\Gamma = \phi \\ 0, & \Omega\cap\Gamma \neq \phi. \end{cases}$$

  - this makes $P_t(\cdot)$ work well with Y.
  - Need lemma: If Y is Koszul or Taylor and if $1 \le l < k \le |\Omega|$, then

$$P_l(P_k(f_\Omega)) = -P_{k-1}(P_l(f_\Omega)).$$

- Want to weaken the restrictions on Y
  - Can Y just be simplicial? What about cellular?
    - So far, examples seem likely but you can always define products (inductively) to satisfy the Leibniz rule but can't always guarantee associativity.
  - What if $Y \cong \widetilde{Y}/W$ where $\widetilde{Y}$ is Taylor or Koszul and $W \subseteq \widetilde{Y}$ is a DG-ideal? It follows that $X * W$ is a DG-ideal of $X * \widetilde{Y}$; what is the relationship between